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# Two Operations for "Ratios" and "Differences" of Distances on the Mental Map

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Ss judged "ratios of distances" and "differences of distances" between pairs of U.S. cities. Results fit the theory that Ss used two comparison processes as instructed. A ratio scale of distances between cities was constructed from the 2 rank orders. From this scale, an interval scale of the city locations on an east–west continuum was derived. This scale agrees with the subtractive model fit to "ratios" and "differences" of easterliness and westerliness, and it also agrees with multidimensional scaling of judged distances between cities. These findings are consistent with the theory that Ss use subtraction when instructed to judge either "ratios" or "differences," but that they can use both ratio and difference operations when the stimuli (in this case, distances) constitute a ratio scale on the subjective continuum.

For a variety of continua, judgments of "ratios" and "differences" of subjective magnitudes appear to be monotonically related (Birnbaum, 1978, 1980, 1982; Birnbaum & Elmasian, 1977; Birnbaum & Veit, 1974; De Graaf & Frijters, 1988; Elmasian & Birnbaum, 1984; Hardin & Birnbaum, in press; Schneider, 1982; Schneider, Parker, & Upenieks, 1982; Veit, 1978, 1980). Because actual ratios and differences would not be expected to be monotonically related in the factorial designs used in this research (Krantz, Luce, Suppes, & Tversky, 1971; Miyamoto, 1983), such experiments provide nontrivial tests of Torgerson's (1961) hypothesis that subjects perceive or appreciate but a single relation between a pair of stimuli, despite instructions to judge "ratios" or "differences."[1]

Torgerson (1961) concluded that if subjects used only one operation for the comparison of two stimuli, then it would be impossible to discover empirically whether that operation is actually a ratio or a difference. And if the operation cannot be determined, then it is impossible to determine how to measure subjective value or to make an empirical comparison of Fechner's logarithmic psychophysical law with Stevens's power law. Torgerson concluded that choices between ratio or subtractive theories and between the power function or logarithmic function for scale values would be decisions, not discoveries.

However, Birnbaum (1978, 1979, 1980, 1982) noted that even if Torgerson's hypothesis is correct for pair judgments (i.e., "ratios" and "differences"), it may be possible to differentiate theories in a wider realm of experiments (e.g., "ratios of differences" and "differences of differences"). If certain of these theories are compatible with data from the wider realm, then an empirical distinction between Ratio and Subtractive Theories for the comparison process may be possible. The theory that has so far been most successful in providing a coherent account of these broader experiments is Birnbaum's (1978) Subtractive Theory. Subtractive Theory can be tested against other theories, including the theory that subjects have the ability to compute only one relation, by means of experiments that involve comparisons of stimulus relations.

## Subtractive Theory Versus Indeterminancy Theory

Indeterminacy Theory states that subjects possess the "mental machinery" to use only one operation to compare subjective values or stimulus relations. On the other hand, Subtractive Theory assumes that subjects can use both subtractive and ratio operations when the subjective scales permit meaningful computations. However, Subtractive Theory allows that many psychophysical continua, such as the pitch or loudness of tones, heaviness of lifted weights, darkness of gray papers, and so on, may be regarded as inherently no more than interval scales. On interval scales, subjects cannot meaningfully calculate ratios, so they compute differences.

For example, experiences of heaviness of lifted weights are represented in Subtractive Theory as positions along a line that does not have a specified zero point. Thus, subjective heaviness experiences are analogous to positions of cities on

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[1] Quotation marks are used to designate instructions to the subject and judgments given by a subject when so instructed. Quotation marks are not used for arithmetical or theoretical statements. For example, when a subject is instructed to judge "ratios," the ratio model may or may not describe "ratio" judgments. Capitalization is used to denote theories that have implications for several experiments; models fit to single arrays of data are not capitalized. For example, Subtractive Theory implies that the subtractive model can be fit to both "ratio" and "difference" judgments and that subjects can use both ratio and difference operations when subjects are instructed to judge "ratios of distances" and "differences of distances."

a mental map. Although physical measures of weight have a well-defined zero point and it is meaningful to discuss the ratio of two physical weights, the ratio of two subjective heaviness values is not meaningful, by the definition of Suppes and Zinnes (1963). The subject, despite lifting a physical weight of zero, uses effort to "lift" nothing and will grow tired "lifting" a zero weight. Thus, Subtractive Theory does not assume a priori that a physical weight of zero corresponds to a sensation of zero but allows the possibility that lifting a physical weight of zero might produce a nonzero sensation of heaviness.

In Subtractive Theory, then, subjective scale values are analogous to positions along a line, on which intervals are defined but ratios are not. Subtractive Theory postulates that when subjects are asked to judge "ratios" of subjective magnitudes on such interval scales, they actually compute differences between subjective values and use the example responses as a category scale.

If the examples for a "ratio" task are geometrically spaced, then the numbers used by the subjects can actually fit a ratio model, even though subjects estimate differences (Birnbaum, 1978, 1980, 1982). Furthermore, when the largest example response mentioned in the instructions is varied, subjects are willing to call the same physical ratio "4," "8," "32," or even "64" (Hardin & Birnbaum, in press; Mellers & Birnbaum, 1982; Mellers, Davis, & Birnbaum, 1984).

On an interval scale, however, ratios of intervals (or distances between stimuli) are meaningful because a zero distance is defined even when a zero subjective value is not. For example, it may not be meaningful to ask, "What is the ratio of the easterliness of Philadelphia to that of San Francisco?", but it is meaningful to ask, "What is the ratio of the distance from San Francisco to Philadelphia relative to the distance from San Francisco to Denver?"

## Judgments on the Mental Map

In order to investigate the analogy between psychophysical judgments and mental maps, Birnbaum and Mellers (1978) asked subjects to perform four tasks: to judge the "ratios" and "differences" of the easterliness and westerliness of U.S. cities. Their results for these four tasks involving the so-called "metathetic" continuum of position were consistent with results for such "prothetic" continua as loudness and heaviness (Birnbaum & Elmasian, 1977; Birnbaum & Veit, 1974; Mellers et al., 1984).

Subjects did not balk at the (apparently) meaningless task of judging "ratios" of easterliness, nor did they insert zero points and compute ratios of distances. Instead, subjects judged that Philadelphia is about "eight times" as easterly as San Francisco; they used the most extreme example in the instructions for the greatest distance in the study. In fact, the ratio model achieved a reasonable fit to these (seemingly) meaningless "ratio" judgments. However, the ratio model led to two inconsistent scales for easterliness and westerliness that did not resemble the actual map.

On the other hand, all four data matrices for the four tasks could be reproduced using a single scale of position, assuming that subjects actually used subtraction to compute either a "ratio" or a "difference." Birnbaum and Mellers (1978) therefore concluded that subjects use subtraction to judge both "ratios" and "differences" of easterliness and westerliness, using a single mental map.

This study compares the Subtractive Theory with Indeterminancy Theory for "ratios" and "differences" of distances. Indeterminancy Theory, as well as Ratio Theory and Eisler's (1978) Transformation Theory, imply that "ratios" and "differences" of distances between stimuli should produce the same rank order, consistent with the use of one operation for both tasks (Birnbaum, 1978, 1979; Eisler, 1978). In contrast, the Subtractive Theory implies that these two tasks should generate two distinct rank orders, connected by ratio and difference operations on a single scale of distances (Birnbaum, 1978, 1982, in press; Veit, 1978). These contrasts among the theories are listed in Table 1, which shows the predicted models for each of four tasks and the scale convergence relations.

Subtractive Theory implies that the scale of city locations derived from the ratio of distances model applied to "ratios of distances" should agree with the scale derived from the difference of distances model applied to "differences of distances," and this scale should in turn agree with the subtractive interpretation of "ratios" and "differences" of easterliness and westerliness (Birnbaum & Mellers, 1978). The second study also compares these scales with the east–west dimension obtained in multidimensional scaling of judged distances between cities.

## Method

### *Experiment 1: "Ratios" and "Differences" of Distances*

Each subject in Experiment 1 performed two tasks, judging "differences" and "ratios" between pairs of distances, in which each distance was designated by two cities. For example, what is the ratio of the distance from San Francisco to Philadelphia relative to the distance from San Francisco to Salt Lake City?

### *Instructions*

On each trial, the subject was presented with four cities (two pairs of cities). Instructions for the "difference" task read (in part) as follows:

> Your task is to judge the difference between two distances—the distance between the first two cities minus the distance between the second two cities. . . . The scale ranges from 80 (the first distance is very very much larger than the second distance) to −80 (the first distance is very very much smaller than the second distance). Zero means the two distances are the same. Feel free to use any integers between −80 and 80.

Nine categories were defined as a guide for the "difference" task: −80, −60, −40, −20, 0, 20, 40, 60, and 80. These values were given category labels varying from 80 = *very very much larger,* 60 = *very much larger,* 40 = *much larger,* 20 = *slightly larger,* 0 = *equal,* . . . to −80 = *very very much smaller.*

Table 1
*Selected Theories of Distance Comparisons*

| Task | Model = Task | Subtractive | Indeterminancy | Transformation |
|---|---|---|---|---|
| | Theory | | | |
| "Differences" | $s_j - s_i$ | $s_j - s_i$ | $s_j - s_i$ | $s_j - s_i$ |
| "Ratios" | $s_j/s_i$ | $s_j - s_i$ | $s_j - s_i$ | $t_j/t_i$ |
| "Differences of distances" | $d_{ij} - d_{kl}$ | $d_{ij} - d_{kl}$ | $d_{ij} - d_{kl}$ | $T[d_{ij}/d_{kl}]$ |
| "Ratios of distances" | $d_{ij}/d_{kl}$ | $d_{ij}/d_{kl}$ | $d_{ij} - d_{kl}$ | $d_{ij}/d_{kl}$ |

*Note.* $s_i$, $s_j$, $s_k$, $s_l$ are subjective values of stimuli; $t = \exp(s)$; $T$ is the log function. If the cities are unidimensional, then $d_{ij} = |s_j - s_i|$. Subtractive theory, unlike Indeterminancy Theory or Transformation Theory, predicts that "ratios of distances" and "differences of distances" are governed by two distinct operations. Ratio Theory (not listed) is ordinally equivalent to Indeterminancy Theory for these tasks, replacing subtraction with division, but it could be distinguished by other experiments (Birnbaum, 1978, 1979).

Instructions for the "ratio" task read (in part) as follows:

> Your task is to judge the ratio of the distance between the first two cities relative to the distance between the second two cities. . . . Feel free to use any number from zero (0) to as large as you need to represent your judgments of the ratios of distances. Remember, 100 means the two distances are the same.

Seven example "ratios" were presented as a guide for the "ratio" task, ranging from 12.5 (the first distance is 1/8 the size of the second distance) to 800 (the first distance is 8 times larger than the second distance). The examples were 12.5, 25, 50, 100, 200, 400, and 800. For both tasks, the example responses were defined without reference to any particular stimuli.

## Stimuli and Designs

Identical stimuli and designs were used for both tasks. The seven cities were San Francisco, California (SF); Salt Lake City, Utah (SLC); Denver, Colorado (Den); Kansas City, Kansas (KC); Champaign-Urbana, Illinois (CU); Columbus, Ohio (Col); and Philadelphia, Pennsylvania (Phil). Figure 1 shows the locations of the cities and the distances used. These cities, which were previously studied by Birnbaum and Mellers (1978), have approximately the same latitude and vary mostly in longitude. The westernmost city was printed first in each pair of distinct cities forming a distance. There were 52 pairs of distances that were constructed from the union of three overlapping designs.

The first design was based on the six intercity distances between the following four cities: San Francisco (SF), Salt Lake City (SLC), Champaign (CU), and Philadelphia (Phil). These six distances were paired with each other to form a complete, 6 × 6, First Pair (distance) × Second Pair (distance), factorial design. The six actual distances (in miles) are 599, 708, 1,267, 1,843, 1,932, and 2,521, for SF–SLC, CU–Phil, SLC–CU, SF– CU, SLC–Phil, and SF–Phil, respectively. The distances (city pairs) are shown by curves in the upper portion of Figure 1. This design allows a test of the theories of comparison operations, and it also permits a test of unidimensionality of these cities.

The second design was based on the six distances formed by the six possible SF–(other city) pairs. These distances are illustrated by the curves in the lower portion of Figure 1. The second design was chosen to permit scaling of the seven cities used by Birnbaum and Mellers (1978), using a subset of the possible 6 × 6 design. Only the first row (each of the six distances vs. SF–SLC), first column (SF–SLC vs. each of the six distances), last row (each of the six distances vs. SF–Phil), and last column (SF–Phil vs. each of the six distances) were presented. (Only 12 additional trials are required because 8 of these trials already occurred in the first design.) The six actual distances are 599, 956, 1,498, 1,843, 2,133, and 2,521, for SF–SLC, SF–Den, SF–KC, SF–CU, SF– Col, and SF–Phil, respectively.

Finally, four additional trials were included in which the first distance was zero (Den–Den or Col–Col) and the second distance was either the smallest nonzero distance (SF–SLC) or the largest distance (SF–Phil). These trials were included to examine how subjects treated zero distances under "difference" and "ratio" instructions.

## Procedure

Instructions, warm-ups, and experimental trials were printed in a separate booklet for each task. For the first task of each session, the experimenter read the instructions and four sample trials to the subjects while the subjects followed along in their booklets. For the second task, the subjects read the instructions on their own. About half the subjects performed the "difference of distances" (*DD*) task first followed by the "ratio of distances" (*RD*) task, and half did the tasks in the opposite order.

After the instructions and samples, each subject made judgments on eight practice trials. The experimenter checked the sample trials to assess the subject's understanding of the task. One such practice trial was Denver to Philadelphia versus Denver to Philadelphia. Because the first two cities are the same as the second two cities, the responses for the "ratio" and "difference" tasks should be 100 and 0, respectively.

The 52 experimental trials were printed in random order, with the restriction that successive trials not share a common pair of cities.

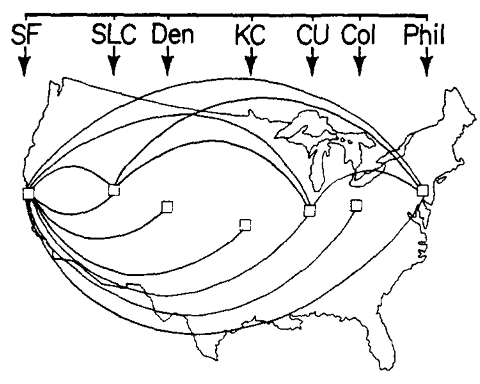


*Figure 1.* Map of the United States, showing locations of cities used as stimuli in Experiment 1. (Upper and lower curved lines indicate city pairs [distances] used in different designs. SF = San Francisco; SLC = Salt Lake City; Den = Denver; KC = Kansas City; CU = Champaign-Urbana; Col = Columbus; and Phil = Philadelphia.)

### Subjects

The subjects were 104 undergraduates who received extra credit in lower division psychology courses at the University of Illinois at Urbana–Champaign.

## Experiment 2: Multidimensional Scaling

### Instructions

The subjects' task was to judge distances between pairs of 15 U.S. cities, including the 7 of Experiment 1. The subjects were given a list of 15 cities with the states in which they were located and were asked to imagine the locations of the cities. The instructions read, in part, as follows:

> Your task is to judge the distance between the two cities. As a scale for making your distance judgments, take 100 to be the distance between San Francisco and Philadelphia. ... In each case, judge the distance between two cities relative to the distance between San Francisco and Philadelphia. For example, if you think the distance between two cities is half the San Francisco–Philadelphia distance, then your judgment of that distance should be 50. If you think the distance is larger than the San Francisco–Philadelphia distance, then your judgment of that distance should be larger than 100.

### Stimuli

The 15 cities were the 7 cities of Experiment 1, plus Seattle, Washington (SE); Los Angeles, California (LA); San Antonio, Texas (SA); Minneapolis, Minnesota (MN); New Orleans, Louisiana (NO); Atlanta, Georgia (Atl); Miami, Florida (MI); and Boston, Massachusetts (BO). The 8 new cities were chosen to produce variation on a second dimension.

### Procedure

After the instructions, the subjects had 10 practice trials that included each city at least once. After the experimenter checked whether the subjects understood the task, each subject then made distance judgments on 108 trials. The first 3 of these 108 trials were unlabeled extra warm-ups, followed by all 105 (i.e., $15(15 - 1)/2$) possible pairs of cities, presented in random order with the restriction that successive trials not share a common city.

### Map Location Task

After the distance judgment task, the subjects were given a list of the 15 cities and an outline of the continental United States showing only the state boundaries. They were instructed to indicate the location of each city by marking and labeling the locations on the map. This task tested each subject's knowledge of the geographical location of the cities (Birnbaum & Mellers, 1978).

### Subjects

The subjects were 39 undergraduates from the same population as in Experiment 1. All the subjects were from Illinois except for two, who were from Indiana and Washington, D.C. Two additional subjects were tested, but their data were excluded from analysis because they placed fewer than nine cities in their proper states in the map location task.

# Results

## Experiment 1: "Ratios" and "Differences"

### One Operation or Two?

The first issue to consider is whether "ratios" and "differences" of distances between pairs of cities are governed by two operations or only one. The one-operation theories (Indeterminancy Theory and Transformation Theory in Table 1 and Ratio Theory) imply that the rank orders of both sets of judgments should be identical. Subtractive Theory (which says that subjects are using two operations, ratio and subtraction, to compare distances) in this case implies that the rank orders will differ systematically between the two matrices. The two-operation theory can be written as follows:

$$\hat{R}_{ijkl} = M_R(d_{ij}/d_{kl}) \qquad (1)$$

$$\hat{D}_{ijkl} = M_D(d_{ij} - d_{kl}) \qquad (2)$$

where $\hat{R}_{ijkl}$ and $\hat{D}_{ijkl}$ are predicted "ratio" and "difference" rank orders; $M_R$ and $M_D$ are strictly increasing monotonic functions that simply rank order the theoretical ratios and differences; $d_{ij}$ and $d_{kl}$ are the subjective distances between cities $i$ and $j$ and $k$ and $l$, respectively. Equations 1 and 2 imply that the two types of judgments will not be monotonically related. For example, $2/1 > 5/3$, but $2 - 1 < 5 - 3$. They also imply that zero distances will be treated differently.

*Zero distances.* The ratio of zero to any positive distance is zero ($0/d = 0$ for all $d > 0$). The difference between zero and some distance depends on that distance ($0 - d = -d$). Table 2 shows that the "distances" of Den–Den or Col–Col behave like proper zero points in the two respective operations. Median "ratios" are all zero, whereas median "differences" are either −40 or −80, depending on whether the distance subtracted from zero is SF–SLC or SF–Phil. These judgments are consistent with two operations. However, it seems possible that the subjects may treat these trials as special cases, so it is necessary to examine the other data on this issue.

*Rank orders.* Not counting the 4 cells involving zero distances, and excluding the six trials involving comparisons of a distance with itself, there are 42 cells in the design. These cells were rank-ordered for median "ratios" and "differences." According to the one-operation theories (Indeterminancy and Transformation; see Table 1), each cell should have the same

Table 2
*Median Judgments for Test of Zero-Point Properties*

| | Ratios of Distances | | Differences of Distances | |
|---|---|---|---|---|
| | First Distance | | First Distance | |
| Second Distance | Den–Den | Col–Col | Den–Den | Col–Col |
| SF–SLC | 0 | 0 | −40 | −40 |
| SF–Phil | 0 | 0 | −80 | −80 |

*Note.* Den = Denver; Col = Columbus; SF = San Francisco; SLC = Salt Lake City; and Phil = Philadelphia.

rank order in both matrices. Instead, 35 of the 42 cell medians changed rank order from one matrix to the other.

To assess whether these changes in rank order are systematically predicted from the Subtractive Theory (two operations), we conducted two ordinal analyses. The first analysis used the Birnbaum and Mellers (1978) subtractive model scale values to calculate predictions for the present experiment. (These scale values had been estimated from the application of the subtractive model to "ratios" and "differences" of easterliness and westerliness, using a single scale to fit all four sets of data.) The second analysis was based on scale values that were fit to the present data (and should therefore yield better fits than predictions based on previous experiments).

The Birnbaum and Mellers (1978) scale values ($s_i$) for the cities are 0, 1, 1.9, 3.2, 4.15, 5.3, and 6.5 for SF, SLC, Den, KC, CU, Col, and Phil, respectively. The distances between pairs of cities were calculated as absolute values of differences in scale value, $d_{ij} = |s_i - s_j|$. Let $R$ and $D$ represent the rank orders of the 42 median "ratios of distances" and "differences of distances," respectively. Let $\hat{R}$ and $\hat{D}$ be the rank order of the predictions of the two-operations theory, given by Equations 1 and 2 (i.e., $\hat{R}$ and $\hat{D}$ are the rank orders of calculated ratios and differences using the Birnbaum and Mellers values to calculate distances).

There are three rank-order correlations of interest, $R$ with $\hat{R}$, $D$ with $\hat{D}$, and $R - D$ with $\hat{R} - \hat{D}$. The first two correlations describe how well the Birnbaum and Mellers (1978) scale values, combined with Subtractive Theory (two operations), predict the rank orders of the medians. The third correlation tests one-operation theories (Indeterminancy and Transformation) against this version of two-operation theory. According to one-operation theories, $R - D$ should all be zero ($\hat{R} - \hat{D} = 0$), and therefore any changes in rank order should be due to chance and unpredictable from two-operation theory. The three Spearman rank-order correlations are .963, .987, and .526, respectively, all significant at the $p < .01$ level. The third correlation rules out Indeterminancy, Transformation, and Ratio Theories in favor of the two operations of Subtractive Theory because one-operation theories require that this correlation must be zero.

One can achieve better fits by estimating the subjective distances from the data and by using estimates of distances to calculate $\hat{R}$ and $\hat{D}$. This approach is described in more detail in the next section. For comparison with the above results, the fitted values yield Spearman correlations of .990, .993, and .839 for the rank orders of $R$ with $\hat{R}$, $D$ with $\hat{D}$, and $R$-$D$ with $\hat{R}$-$\hat{D}$, respectively.

The third correlation, $R - D$ with $\hat{R} - \hat{D}$, again shows that differences in rank order between two tasks are systematically predicted by the two-operation theory. Therefore, the data require rejection of the one-operation theories (Table 1) in favor of Subtractive Theory, which assumes that subjects use both ratio and difference operations to compare distances.

For example, the median "ratio" of (SF–CU)/(SF–SLC) exceeds (SF–Phil)/(CU–Phil) (300 vs. 250), whereas the median "differences" have the opposite order (40 vs. 50). It is interesting to note that this pattern is predicted by two-operation theory applied to the subtractive model scale values of Birnbaum and Mellers (1978), but it would not be predicted from the physical distances. Of 104 subjects, 67 said the former "ratio" was greater than or equal to the latter, whereas 69 said the "difference" of (SF–CU)–(SF–SLC) was less than (SF–Phil)–(CU–Phil). Taken together, the data refute the theory that subjects compare distances by the same operation; instead, the data show systematic evidence of two operations.

## *Fit of Two-Operation Theory*

*Metric Fit.* A metric form of two-operation theory can be written as

$$\hat{RD}_{ijkl} = a_R(d_{ij}/d_{kl}) + b_R \quad (3)$$

$$\hat{DD}_{ijkl} = a_D(d_{ij} - d_{kl}) + b_D \quad (4)$$

where $\hat{RD}_{ijkl}$ and $\hat{DD}_{ijkl}$ are the predicted "ratio" and "difference" of the distance between cities $i$ and $j$ relative to the distance between cities $k$ and $l$. The subjective distances are $d_{ij}$ and $d_{kl}$ respectively, and $a_R$, $b_R$, $a_D$, and $b_D$ are constants, to be estimated from the data.

If these cities can be represented as points on a line, then the sum of the distances from SF–SLC plus SLC–CU plus CU–Phil should equal the distance from SF–Phil. To test the unidimensionality of these cities, two versions of the two-operation model were fit to the median judgments. One version constrains the distances to be unidimensional; this constraint can be expressed as follows:

$$d_{ij} = |s_j - s_i|, \quad (5)$$

where $s_j$ and $s_i$ are scale values to be estimated, representing subjective positions of the cities on an east–west continuum. The "unconstrained" version of the model places no restrictions on the distances to be estimated.

The models were fit to the median judgments by using a computer program written to select parameters for Equations 3 and 4 to minimize an index of fit.[2] The minimization was accomplished with the aid of Chandler's (1969) subroutine using the method of Birnbaum (1980). The unconstrained model achieved a fit of .051. The constrained (unidimensional) model used Equation 5, which requires three fewer parameters, and achieved a fit of .056, almost the same as the fit of the more general model. Examination of the distances estimated by the unconstrained fit and of the pattern of residuals for the constrained model indicated that the unidimensional model can be retained for these seven cites (see also the Appendix).

[2] The index of fit, $L$, is defined as follows:

$$L = \Sigma(DD_i - \hat{DD}_i)^2/\Sigma(DD_i - \bar{DD})^2 + \Sigma(r_i - \hat{r}_i)^2/\Sigma(r_i - \bar{r})^2$$

where $L$ is the index to be minimized, $DD_i$ is the median "difference of distance" response, $\bar{DD}$ is the mean over cells, and $\hat{DD}_i$ is the predicted value. The symbols $r_i$, $\hat{r}_i$, and $\bar{r}$ are log "ratio of distance," log of predicted "ratio of distance," and mean log "ratio of distance," respectively. The summation is over 48 cells in the experimental design, omitting the four trials involving zero distances (in Table 2). For further information on the fitting procedure, see Birnbaum (1980).

*Ordinal Fit.* The two-operation theory with constrained distances can be modified to allow for nonlinear relationships between subjective values and overt judgments as follows:

$$\hat{RD}_{ijkl} = J_{RD}[\,|s_j - s_i|\,/\,|s_l - s_k|\,] \qquad (6)$$

$$\hat{DD}_{ijkl} = J_{DD}[\,|s_j - s_i| - |s_l - s_k|\,] \qquad (7)$$

where $J_{RD}$ and $J_{DD}$ are strictly monotonic judgment functions, and $\hat{RD}$ and $\hat{DD}$ are predicted judgments of "ratios of distances" and "differences of distances." These judgment functions were estimated by successively transforming the median judgments to improve the fit of the constrained solution, using a method similar to that of Cliff (1972). After two iterations, the lack-of-fit index was .012, and the results were deemed good enough to stop. (For comparison, the unconstrained, multidimensional model fit the transformed scores only slightly better, .011.)

The estimated $J_{DD}$ and $J_{RD}$ functions are shown in Figures 2 and 3. The functions are nearly linear in both cases. Here is a case in which the subjects are presumed to be actually using a ratio operation (unlike the presumption for simple "ratios" of heaviness, loudness, easterliness, or westerliness), and in this case, $J_{RD}$ is found to be nearly linear. Previous research with simple "ratios" concluded that "ratio" judgments are a positively accelerated function of subjective value (Birnbaum, 1980); however, in those studies it is theorized that the subjective values are governed by subtraction. The Appendix reports additional analyses that indicate that the $J$ functions in these studies are not well represented as power functions.

The fit of the transformed medians to the two-operation, unidimensional constrained model is shown in Figure 4. Points in the upper panels show data for "ratios"; points in the lower panels show "differences." Lines show predictions

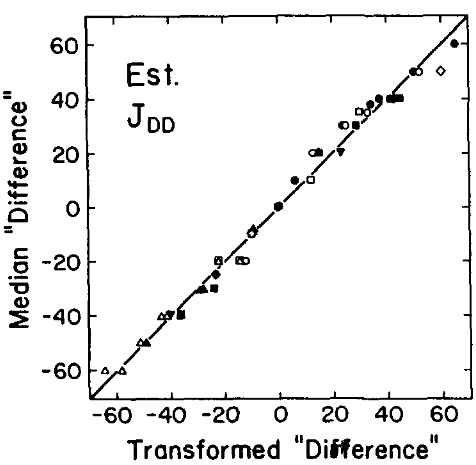


*Figure 2.* Estimated judgment function for "differences of distances" task in Experiment 1. (● = SF–SLC [San Francisco to Salt Lake City]; ○ = CU–Phil [Champaign–Urbana to Philadelphia]; ■ = SLC–CU [Salt Lake City to Champaign–Urbana]; □ = SF–CU [San Francisco to Champaign–Urbana]; ▲ = SLC–Phil [Salt Lake City to Philadelphia]; △ = SF–Phil [San Francisco to Philadelphia]; ◆ = SF–KC [San Francisco to Kansas City]; ◇ = SF–Den [San Francisco to Denver].)

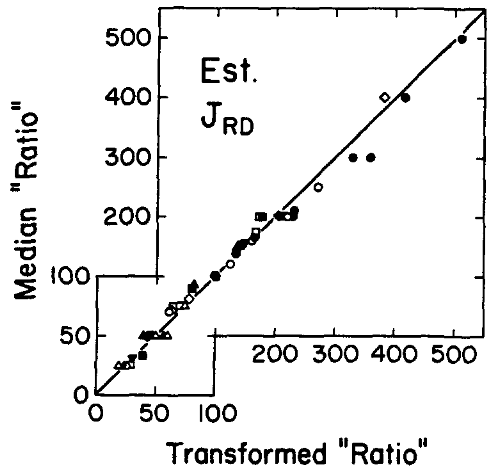


*Figure 3.* Estimated judgment function for "ratio of distances" task in Experiment 1. (Scales have been doubled for "ratios" of less than 1. ● = SF–SLC [San Francisco to Salt Lake City]; ○ = CU–Phil [Champaign–Urbana to Philadelphia]; ■ = SLC–CU [Salt Lake City to Champaign–Urbana]; □ = SF–CU [San Francisco to Champaign–Urbana]; ▲ = SLC–Phil [Salt Lake City to Philadelphia]; △ = SF–Phil [San Francisco to Philadelphia]; ◆ = SF–KC [San Francisco to Kansas City]; ◇ = SF–Den [San Francisco to Denver].)

based on a single scale of city locations fit to all of the data simultaneously. The left panels in each case show results for the full 6 × 6 design; the right panels show the results for the partial, 6 × 6, SF–city design. The estimated values of the cities were as follows: SF = 0 and SLC = 1 were fixed arbitrarily, and the estimated values for the others were Den = 1.38, KC = 2.50, CU = 3.45, Col = 3.72, and Phil = 5.35. The estimated constants (Equations 3 and 4) were as follows: $a_D$ = 14.91 and $b_D$ = 0.67, for "differences"; and $a_R$ = 95.51 and $b_R$ = 4.78, for "ratios" (see Figures 2 and 3).

The transformed medians appear very close to predictions. The Spearman correlations of predictions with obtained values exceed .99 for both "ratios" and "differences." This theory predicts that 37 of the 42 nonzero, nondiagonal cells will change rank order from "ratio" to "difference" matrices. Of these 37, 32 cells changed in the predicted directions, whereas only 5 cells changed in the opposite direction. The theory correlates .839 with the magnitudes of these changes in rank order. Therefore, the data appear ordinally consistent with the constrained, two-operation theory.

## *Scale Convergence*

The two-operation Subtractive Theory of *RD* and *DD* leads to a ratio scale of distances, which (for the constrained version) leads in turn to an interval scale of the city locations. The next question is to ask how the derived scale of position compares with scales that would be obtained from rival models applied to "ratios" and "differences" of easterliness and westerliness, from Birnbaum and Mellers (1978).

Birnbaum and Mellers (1978) noted that the rank order of four matrices, "ratios" and "differences" of easterliness and westerliness, could be reproduced by a single scale using the subtractive model. That subtractive model scale is shown on the abscissa of Figure 5. The triangles in Figure 5 show scales derived from the ratio model applied to easterliness and westerliness judgments by Birnbaum and Mellers. Ratio model scales of easterliness and westerliness are nonlinearly related to each other. The new scale values, derived from two operations for *RD* and *DD* are shown as solid circles in Figure 5. There appears to be reasonable agreement (broken line) between the present scale of position and the subtractive model scale of Birnbaum and Mellers.

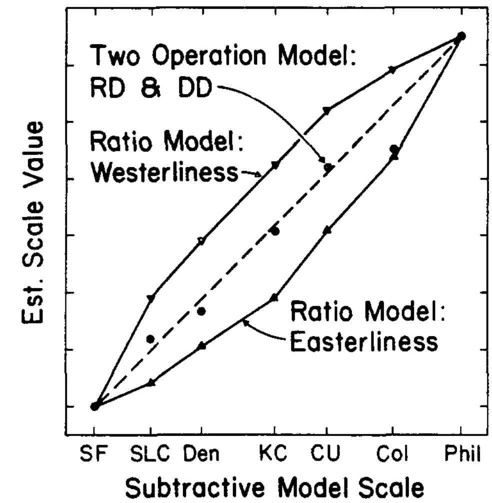


*Figure 5.* Comparison of subjective scales of location based on different theories. (Abscissa is spaced according to subtractive model scale of Birnbaum and Mellers [1978]. Scale based on two-operation theory of "ratios of distance" and "differences of distance" for Experiment 1 agrees with subtractive model scale [dots]. Agreement suggests Subtractive Theory can use one scale of position to explain "ratios" and "differences" of easterliness and westerliness, "ratios of distances" and "differences of distances.")

## *Experiment 2: Multidimensional Scaling*

Because physical maps of the United States usually represent locations on a two-dimensional plane, it seems reasonable that subjective or cognitive maps of cities varying in two physical dimensions might also be two dimensional (Kruskal & Wish, 1978).

The euclidean distance function can be written as follows:

$$d_{ij} = \{\Sigma \mid s_{ip} - s_{jp} \mid^2\}^{1/2} \quad (8)$$

where $s_{ip}$ and $s_{jp}$ are coordinates of cities *i* and *j* on dimension *p*. The computer program KYST (Kruskal, Young, & Seery, 1977) was used to fit the euclidean model to the data, allowing a monotonic function between judged distances and $d_{ij}$. As a function of the number of dimensions, stress values were .17, .03, .02, and .01 for 1, 2, 3, and 4 dimensions, respectively, which seem to indicate a two-dimensional solution with stress of .03. The monotonic function relating distances (subjective values) to distance judgments (data) was slightly positively accelerated.

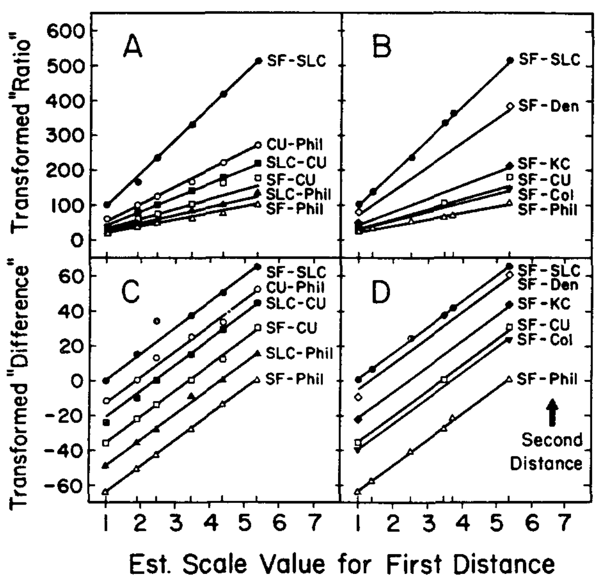


*Figure 4.* Fit of two-operation model (Subtractive Theory) to "ratios" and "differences" of distances for Experiment 1. (Points in upper panels show median transformed "ratios"; points in lower panels show median transformed "differences." Lines show predictions of two-operation model constrained so that cities are unidimensional. The same parameters are used for predictions in all four panels. SF–SLC = San Francisco to Salt Lake City; CU–Phil = Champaign-Urbana to Philadelphia; SLC–CU = Salt Lake City to Champaign-Urbana; SF–CU = San Francisco to Champaign-Urbana; SLC–Phil = Salt Lake City to Philadelphia; SF–Phil = San Francisco to Philadelphia; SF–KC = San Francisco to Kansas City; SF–Den = San Francisco to Denver.)

Figure 6 shows the best-fit coordinates ($s_{ip}$) from KYST, superimposed on a map of the United States, rotated and scaled so that San Francisco and Philadelphia are in their appropriate locations. The broken lines connect the actual (solid circles) and subjective locations (open circles) of the cities. *INDSCAL*, a program that fits an individual-differences model (Arabie, Carroll, & DeSarbo, 1987; Carroll & Wish, 1974), was also applied to the subjects' individual judgments, leading to a two-dimensional map for the cities that was very similar to the KYST solution. The close proximities of SF with LA and SLC with Den were also found in the map location task, in which subjects also tended to place these same cities too close together.

The orthogonal projections of seven cities (SF, SLC, Den, KC, CU, Col, and Phil) onto the line between SF and Phil were scaled for comparison with the unidimensional scale values from Experiment 1. These scaled values are plotted in the margin of Figure 6 and are labeled *A*.

In order to see the mathematical effect of the other eight cities in the current study on these seven projections, distances among only these seven cities were scaled by KYST in one dimension, yielding a stress of .006. The resulting scale values, labeled *B* in Figure 6, are almost identical to the scale in A. The scale labeled *C* shows the subjective locations from Experiment 1, based on the two-operation Subtractive Theory applied to judgments of "ratios" and "differences" of distances. Scale D shows the subtractive model scale from Birn-

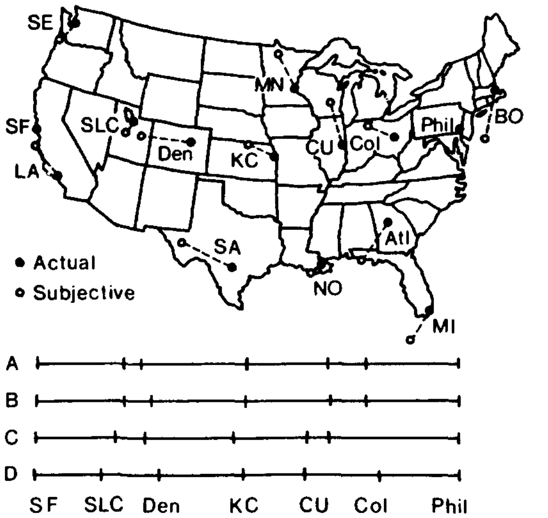


*Figure 6.* Multidimensional scaling solution (Experiment 2) superimposed on a map of the United States. (Actual city locations are shown as solid circles; open circles connected by broken lines show subjective locations, scaled so that San Francisco [SF] and Philadelphia [Phil] correspond to their actual locations. Scale A shows projections on an east–west continuum; Scale B shows a unidimensional solution fit to seven cities only; Scale C is from Experiment 1; Scale D is the subtractive model scale of Birnbaum & Mellers [1978].)

baum and Mellers (1978). Other than Col, which was based on only four judgments in Experiment 1, the scales are very similar. On all subjective scales, the West is compressed relative to the physical scale, the Midwest is relatively expanded, and CU is closer to the center of the United States than its actual location.

## Discussion

### *Indeterminancy Theory Refuted*

The data of Experiment 1 are not consistent with the theory that subjects possess only one operation for comparing subjective values. Instead, the rank orders of the two judgments changed systematically, consistent with the theory that subjects can indeed use both ratio and difference operations when asked to judge "ratios" and "differences" of distances.

The rejection of Indeterminancy Theory allows us to circumvent Torgerson's (1961) conclusion that it might be impossible to discover how subjects compare two stimuli. Because "ratios" and "differences" of distances obey ratio and subtractive models, a ratio scale of distances can be derived. It is then possible to discover empirically the model that generates these distances and to obtain a scale of city locations. The data are consistent with the hypothesis that distances among these seven cities can be represented as intervals on a unidimensional east-west continuum. This scale of position agrees with the Subtractive Theory interpretation of "ratios" and "differences" of "easterliness" and "westerliness," from Birnbaum and Mellers (1978).

### *Subtractive Theory*

The agreement of the present scale values with those of the subtractive model of "ratio" judgments (Figure 5) suggests that when subjects are instructed to judge "ratios" they may or may not use a ratio operation: It appears that subjects use the ratio operation for "ratios of distances" in the present case but that they use subtraction when instructed to judge "ratios" of easterliness or westerliness.

The following simplified theory will give a reasonable approximation to the rank orders of all seven data arrays, including four from Birnbaum and Mellers (1978), two for the present results of Experiment 1, and the data of Experiment 2 using a single scale of city location:

$$RE_{ij} = \exp((s_j - s_i)),$$
$$RW_{ij} = \exp((s_i - s_j)),$$
$$DE_{ij} = (s_j - s_i),$$
$$DW_{ij} = (s_i - s_j),$$
$$RD_{ijkl} = (d_{ij}/d_{kl}),$$
$$DD_{ijkl} = (d_{ij} - d_{kl}),$$

and

$$d_{ij} = \{(s_j - s_i)^2 + (v_j - v_i)^2\}^{1/2},$$

where $RE_{ij}$ and $RW_{ij}$ are "ratios" of easterliness and westerliness, $DE_{ij}$ and $DW_{ij}$ are "differences" of easterliness and westerliness, and $RD_{ijkl}$ and $DD_{ijkl}$ are "ratios of distances" and "differences of distances," respectively. Note that judgment of "ratios" of easterliness and westerliness are exponential functions of algebraic differences in location. The exponential judgment function has no effect on the rank order but is used to explain the numerical pattern of "ratio" judgments (Birnbaum, 1978, 1980; Birnbaum & Mellers, 1978). However, "ratios of distances" are represented as a ratio operation, and the transformation from subjective ratios to judged "ratios of distances" in this case is nearly a similarity transformation (Figure 3). This theory uses a single scale of east–west city locations ($s_i$) and assumes that the second dimension values ($v_i$) are nonzero only for the eight additional cities of Experiment 2.

### *Ratio Theory*

Ratio Theory does not give a coherent account of the data. First, Ratio Theory leads to two scales for judgments of easterliness and westerliness that are reciprocally related (Figure 5). Part of this reciprocal uncertainty is inherent in the ratio model; however, as Birnbaum and Mellers (1978) noted, there is a form of Ratio Theory in which the scales of easterliness and westerliness would have been linearly related. Second, Ratio Theory implies that "ratios of distances" and "differences of distances" should have the same rank order, the order of a ratio of ratios model (Birnbaum, 1978, 1979). Contrary to Ratio Theory predictions, the orders differed.

To save the ratio model for simple "ratios," one would have to represent "ratios of distances" by a complex, exponential of a ratio (Birnbaum, 1978), which then contradicts the theory that people always compute ratios when so instructed. Furthermore, ratio theory would not yield a map that agrees with the multidimensional scaling solution, based on judged distances among cities that varied in two dimensions (Figures 5 and 6).

### *Transformation Theory*

Eisler (1978) proposed interesting solutions to certain difficulties raised for the ratio model posed by Birnbaum (1978). According to Eisler's Transformation Theories, the tasks of judging "ratios" or "differences" involve different subjective scales (for a related development, see also Marks, 1982). Eisler's theories provided a systematic approach that could in principle save the theory that subjects instructed to judge "ratios" actually use a ratio operation, by assuming that they use scales exponentially related to scales involved in "difference" judgments. Eisler developed two theories for the six tasks from premises concerning the models and scales induced by different tasks.

Unfortunately, both versions of Eisler's (1978) theory lead to the implication that "differences of differences" should show the same rank order as "ratios of differences," namely that both rank orders should be that of a ratio of differences model. Eisler noted this difficulty and argued that if subjects indeed were attempting to compute $\log[\log(a/b)/\log(c/d)]$ as dictated by the ratio version of his theory for "differences of differences," when $b > a$, then $\log(a/b)$ would be negative and the second log operation would be impossible. Therefore, he argued, perhaps subjects "reinterpret" the task. However, in the present study, subjects were asked to judge "differences of distances" (rather than "differences of differences"). Under these instructions, the major argument for "reinterpretation" does not apply. Without the "reinterpretation" argument, Eisler's theory leads to the incorrect prediction that the two tasks in the present study should have had the same rank order.

### *Diagnostic Tests*

Rule, Curtis, and Mullins (1981) argued that the evidence on the ratio/difference issue might be equivocal. They expressed concern that for certain stimulus continua, perhaps subjects do use two operations for "ratios" and "differences," but the experiments failed to detect them. They noted that stimulus range, spacing, number of levels, and use of between versus within-subjects designs might make it difficult to make a correct rejection of the null hypothesis of one operation in favor of two operations (see Rule & Curtis, 1980; Veit, 1980).

However, this study used the same cities and the same general procedures and designs as those of Birnbaum and Mellers (1978), but this study found evidence of two operations. Apparently, the key is that in Birnbaum and Mellers (1978), subjects were asked to judge "ratios" and "differences" of easterliness and westerliness, whereas in this study subjects were asked to judge "ratios" and "differences" of distances between cities. Because so many other features of the studies are the same or similar, the present findings diminish the plausibility of the argument that something about the procedures inhibit detection of two operations when subjects actually use them. Parker, Schneider, and Kanow (1975) concluded that subjects use two operations for comparisons of line lengths. By analogy with the present results, it seems reasonable that line segments are perceived as distances between end points. For further discussion, see also Mellers et al. (1984), who provided a detailed study of the issues raised by Rule et al. (1981).

### *Related Research on Comparison Operations*

The present data are consistent with previous research on comparisons of stimulus relations. Veit (1978) found that judgments of "ratios of differences" were consistent with a ratio of differences model, could not be rescaled to fit other simple models, and led to a scale that agreed with the subtractive interpretation of "ratios" and "differences."

Hagerty and Birnbaum (1978) and Birnbaum (1982) asked subjects to compare stimuli or stimulus pairs in six tasks. Both of these studies found that "ratios of differences" and "differences of differences" yielded different rank orders, consistent with the instructed tasks, whereas "ratios of ratios" and "differences of ratios" yielded data that had the same rank order as "differences of differences" and could be fit by the differences of differences model. Furthermore, all six data arrays could be reproduced using the same scale values when Subtractive Theory is assumed (Birnbaum, 1978, 1982, 1983).

### *Cognitive Maps and Analogue Representations*

Although the subtractive model scale values of Birnbaum and Mellers (1978) and the present studies resemble the actual map (Figures 1, 5, 6), there are some important differences. The subjects of both studies, who were tested in Champaign-Urbana (CU), tend to exaggerate nearby distances and place CU too close to the center of the country. Because the subjective map is not a linear transformation of the actual map, subjective distances are not a monotonic function of actual distances. For example, in physical distances, $|\text{SF} - \text{Den}| > |\text{CU} - \text{Phil}|$; however, both Birnbaum and Mellers and the present experiments found that for subjective distances, $d|\text{CU} - \text{Phil}| > d|\text{SF} - \text{Den}|$. Holyoak and Mah (1982) also found similar trends; in their studies, Midwesterners compared distances from the Atlantic or Pacific to cities. In all cases, the Midwest was expanded and the West was compressed relative to the actual maps.

It is important to emphasize that although the present cognitive maps resemble the actual maps, the judged distances are not a monotonic function of actual distances. Recent research on cognitive maps have interpreted such "errors" as evidence that subjects do not use mental maps to judge directions and distances (e.g., Hirtle & Jonides, 1985; Mc-

Namara, 1986; Stevens and Coupe, 1978). It has been argued that subjects store spatial relationships as propositions rather than analogue representations, such as mental maps. However, analogue representations are more flexible than these authors have conceded and can account for phenomena that have been cited as evidence of hierarchical representations.

To illustrate how analogue representations could account for the phenomena reported in the cognitive mapping literature, consider the properties of cognitive maps and of more general analogue representations. First, a cognitive map can contain distortions (Hourihan & Jones, 1979). For example, Shepard (1957) asked subjects to recall the states and used the difference in recall order to construct a "Bostonian's Map of the United States" in which New England is expanded and the South and West are compressed, relative to the actual map. Apparently, Shepard's subjects, who were tested near Boston, tended to recall New England states in more consistent orders than states farther away. Indeed, the present results, Birnbaum and Mellers (1978), and Holyoak and Mah (1982) all found that judged distances among nearby cities were expanded relative to distances among cities farther away. When there are such distortions, then subjects' judgments of distance will be a monotonic function of subjective distances and will not be a monotonic function of physical distances.

Second, distortions in the mental map will also yield errors in judging directions. For example, Stevens and Coupe (1978) found that Reno, Nevada, is judged to be north-east of San Diego, California, even though it is actually north-west. Propositional statements, such as "Nevada is east of California," may perhaps cause changes in a cognitive map, but once changed, the subjective representation might still be a euclidean subjective mental map. On the other hand, these errors might be due instead to perceptual transformations induced by lines and borders. The introduction of borders, lines, streets, clusters, or intervening cities and the tendencies to make lines achieve certain angles may cause changes in the cognitive map analogous to the effects of visual illusions (Birnbaum, 1983; Thorndyke, 1981; B. Tversky, 1981).

Third, although asymmetries in judged distances (e.g., Holyoak & Mah, 1982; Sadalla, Burroughs, & Staplin, 1980) pose difficulties for euclidean representations (A. Tversky, 1977), asymmetries do not rule out analogue representations. For example, Krumhansl (1978) showed that when geometric models are generalized to include density phenomena, extending Parducci's (1973) range-frequency theory to multiple dimensions (Birnbaum, 1974; Parducci, 1982), analogue models can accommodate asymmetry. Thorndyke's (1981) demonstration that distance judgments depend on the density of cities between points fits in nicely with the multidimensional extensions of range-frequency theory (Birnbaum, 1982; Krumhansl, 1978).

To further illustrate how analogue representations can account for asymmetries, consider the following hypothetical experiment: Subjects explore a town by riding a bicycle around town and later judge all distances from point to point. It is conceivable that subjects might use the time to ride from Point A to Point B as a measure of distance. It may easily take less time to ride downhill from A to B than to ride uphill from B to A. Such asymmetry might be explained as a consequence of stored propositions or retrieval processes; however, such asymmetry might also be explained by superimposing an analogue representation of the gravitational effects of hills onto a mental map. Such an analogue representation also seems a reasonable explanation for asymmetries due to landmarks.

In summary, analogue representations that permit a subjective map (rather than require the map to be correct) would provide a useful rival to the hierarchical representations recently proposed. Analogue representations that include density effects (Birnbaum, 1982; Krumhansl, 1978), that allow implicit scaling (Holyoak & Mah, 1982), or that allow the effects of gravity on trip times (given above) would explain asymmetries and may provide a better basis for development of a coherent theory that would explain the convergence of scales from different tasks as in this study.

## Conclusion

In summary, because "ratios" and "differences" of distances are consistent with the two instructed operations, it follows that Indeterminancy Theory, Ratio Theory, and Eisler's (1978) Transformation Theories have received a severe blow. Those theories all imply that "ratios of distances" and "differences of distances" should be monotonically related.

Subtractive Theory provides a coherent account of the present data, and it leads to scales that agree with the results of multidimensional scaling and the subtractive scale of Birnbaum and Mellers (1978). In Subtractive Theory, subjective scales are represented by points on a mental map. Subjects asked to judge "ratios" or "differences" of easterliness or westerliness use the subtractive operation for both tasks because ratios are not meaningful on an interval scale. However, even on an interval scale, ratios and differences of distances are meaningful, and subjects appear to use both ratio and subtractive operations to compare distances.

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*(Appendix follows on next page)*

## Appendix

The assumption that the distances among the seven cities of roughly equal latitude can be represented as intervals on a line allows a test between two theories of the output transformation for magnitude estimation. Rule and Curtis (1982) assumed that the output function for magnitude estimation is a power function that represents the inverse of the psychophysical function for number. They reported an average value of the exponent of 1.47. Birnbaum (1980), however, argued that the relationship between subjective value and overt response is best represented as a judgment function that depends on the stimulus and response range and distribution, and that it can attain forms that violate the power function. For further discussion of this issue, see Krueger (1989) and Birnbaum (1989).

Accordingly, both constrained and unconstrained versions of the two-operation model were fit with a power function in Equation 1 as follows:

$$\hat{R}D_{ijkl} = a_R(d_{ij}/d_{kl})^m + b_R \quad \text{(A1)}$$

where $m$ is fixed to .9, 1.0, or 1.47, or $m$ was free (estimated by the program). Indexes of lack of fit are shown in Table A1.

The solutions of the distances for the unconstrained solutions with $m = 0.9$ and $m = 1.47$ are shown in Figure A1. Figure A1 shows that when $m = 1.47$, the distances are severely subadditive. The distance from SF–Phil is much less than the sum of the distances along the way. Even though subjects say that the distance from SF to Phil is "five times" the distance from SF to SLC, this model interprets this ratio to be only 2.89. (Remember, subjective ratios are raised to the $m$ power in this model.) Therefore, despite having reasonable fits for the unconstrained solution, large values of $m$ must be rejected if these cities are supposed to vary on a single dimension. When the distances are constrained to be unidimensional, the fit of $m = 1.47$ is the worst of the values compared (.097).

For $m = 0.9$ to 1.0, however, the unconstrained distances are nearly unidimensional. As Figure A1 shows, the estimated distances for $m = 0.9$ are approximately additive along a line. When the distances are constrained to be unidimensional, for $m = 0.9$ or 1.0, the fit remains about the same as for the unconstrained cases. In summary, Table A1 shows that seven of the eight variations of the models have an overall index of fit of about .05, but the constrained model with $m \geq 1.47$ fits markedly worse (about .10). Unidimensionality appears to be acceptable for these seven cities, which are roughly on a line on the actual map, and on the two-dimensional, multidimensional scaling solution. If unidimensionality is assumed, then Table A1 and Figure A1 indicate that $m$ is close to 1.

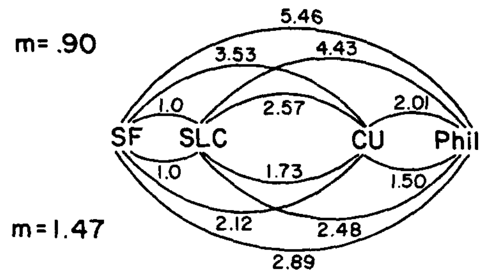


*Figure A1.* Unconstrained solutions for distances in Experiment 1, assuming $m = 0.9$ or $m = 1.47$. (Values are estimated distances from Equations A1 and 4. When $m$ is greater than 1, estimated distances are subadditive, when $m$ is 0.9, distances are nearly additive.)

Table A1
*Lack of Fit for Eight Versions of Two-Operation Model Fit to Median Judgments*

| | *m* fixed | | | *m* free to vary | |
|---|---|---|---|---|---|
| Dimensionality | $m = 0.9$ | $m = 1.0$ | $m = 1.47$ | fit | best-fit $m$ |
| Unconstrained | .054 | .051 | .047 | .046 | $m = 1.751$ |
| Constrained | .054 | .056 | .097 | .054 | $m = .908$ |

*Note.* Index of fit is the sum of proportions of variance of deviations, summed up across two matrices (see Footnote 2). Constrained model uses five parameters for the seven cities, and distances are constrained to be unidimensional. Unconstrained model allows distances to be estimated in the model.

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