Cognitive Biases
J.-P. Caverni, J.-M. Fabre and M. Gonzalez (Eds.)

THEORIES OF BIAS IN PROBABILITY JUDGMENT

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Abstract. When psychologists study human judgments of probability, judged probabilities unfortunately do not conform to the equations of probability theory. Because probability theory offers such a convenient and compelling structure for discussing beliefs about ambiguous and uncertain events, many scholars have found it disturbing to think that humans might have been rational enough to invent probability theory but not rational enough to use it in their daily thought. This chapter will explore explanations of the discrepancies between judged probabilities and the implications of probability theory.

Many decisions in the modern world are based on subjective probabilities. Subjective probabilities guide discussions of critical issues because objective probabilities are often undefined in many situations in which people wish to use the language of probability to plan for future events. For example, the "Star Wars", strategic defense initiative would seem a good idea if it appeared likely to reduce the likelihood of nuclear war or to increase the chances of survival given accidental nuclear attack. On the other hand, some argue that if one side were to develop a defense system that was perceived to be 50% effective, the other side might decide to double their arsenal of weapons to restore the balance of terror. Such increases in arms could increase the likelihood that the weapons would be used and therefore increase the chance of nuclear war and thereby increase the chance that the world will be destroyed by nuclear war. Notice that these arguments involve intuitive probabilities that the weapons will work, that certain actions would be taken by one side or the other, or that an accident might occur. These intuitive probabilities and the decisions they engender will determine the future of the planet, yet they cannot be checked for their accuracy.

With enough replicas of the earth and enough time, perhaps a very powerful experimenter could observe the proportion of earth replicas annihilated by nuclear war with or without any given policy.

Such experiments could in principle provide an empirical basis for decision making. However, in the absence of such experiments, such likelihoods are left to be determined by human judgment. The survival of the planet now rests on the capability of humans to evaluate probability and to make proper decisions in the face of uncertainty.

Unfortunately, when psychologists study human judgments of probability, judged probabilities do not conform to the equations of probability theory. Because probability theory offers such a convenient and compelling structure for discussing beliefs about ambiguous and uncertain events, many scholars have found it disturbing to think that humans might have been rational enough to invent probability theory but not rational enough to use it in their daily thought. This chapter will explore explanations of the discrepancies between judged probabilities and the implications of probability theory.

Probability Theory

Although many different philosophical underpinnings have been proposed for the concept of probability (Krantz, Luce, Suppes, & Tversky, 1971; Kyburg & Smokler, 1964; Savage, 1954; von Winterfeldt & Edwards, 1986), this chapter will consider the "standard" probability theory presented in the introductory texts on statistics and probability (e.g., Hogg & Craig, 1965; Mosteller, Rourke, & Thomas, 1961). Probabilities will be numbers assigned to the beliefs that events will occur, and the same language will be employed for unique events as, "there will be a new champion in boxing this year" and for events that are replicable such as, "five coins will come up heads when ten are flipped." Events are represented by sets (and set algebra), and probability is a measure on the sets, taking values between zero (for the null set) and one (for the universal set), that satisfies the following equations:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad (1)$$

where P(A) and P(B) are the probabilities of events A and B; $P(A \cup B)$ is the probability of the union (either A *or* B, denoted, $\cup$); and $P(A \cap B)$ is the probability of the conjunction, or intersection of events (both A *and* B, denoted, $\cap$). If the intersection is the null set, probability is additive across mutually exclusive events.

The probability of an event, A, and its complement, A' (where A' is the complement of A, or not-A), sum to one, because an event and its complement are defined to be mutually exclusive and exhaustive; hence,

$$P(A) = 1 - P(A'). \qquad (2)$$

If A' is the null set, it is impossible and A is a certainty, so $P(A') = 0$ and $P(A) = 1$.

The probability of the conjunction of events can be written as follows:

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A), \qquad (3)$$

where $P(A|B)$ and $P(B|A)$ are the conditional probabilities of A given B, and of B given A, respectively. The probability of a simple event, A, can also be expressed as follows:

$$P(A) = P(A \cap B) + P(A \cap B'), \qquad (4)$$

because A can either occur with B or without B (there are no other ways), and the intersection of B and B' is empty, by definition.

For two constituent events, A and B, one could ask twenty simple probability questions, including 4 probabilities of the events and their complements [$P(A)$, $P(A')$, $P(B)$, $P(B')$], 4 conjunctions [$P(A \cap B)$, $P(A \cap B')$, $P(A' \cap B)$, $P(A' \cap B')$], 4 unions [$P(A \cup B)$, $P(A \cup B')$, $P(A' \cup B)$, $P(A' \cup B')$], and 8 conditionals [$P(A|B)$, $P(A|B')$, $P(A'|B)$, $P(A'|B')$, $P(B|A)$, $P(B'|A)$, $P(B|A')$, $P(B'|A')$].

However, among these 20 probabilities, there are only 3 degrees of freedom, because once three values are known [for example, $P(A)$, $P(B)$, and $P(A \cap B)$], the remaining 17 can be calculated from the equations. Furthermore, even these three values are constrained, because $P(A \cap B)$ must be less than or equal to $P(A)$, less than or equal to $P(B)$, and greater than or equal to $P(A) + P(B) - 1$.

For example, if $P(A)$ is .7 and $P(B)$ is .6, then $P(A') = .3$ and $P(B') = .4$, by Equation 2. From Equations 3, $P(A \cap B)$ must be less than or equal to .6. Additionally, $P(A \cap B)$ must also be greater than or equal to .3 [by Equation 4, $P(A \cap B) + P(A \cap B') = .7$ and $P(A \cap B') + P(A' \cap B') = .4$; therefore, $P(A \cap B) = .3 + P(A' \cap B')$, which is greater than or equal to .3 because $P(A' \cap B') \geq 0$]. This constraint also implies that $P(A|B) \geq .5$, since $P(A|B) = P(A \cap B)/P(B)$. Suppose

$P(A \cap B) = .4$; from Equation 4, it follows that $P(A \cap B') = .3$, $P(A' \cap B) = .2$, and $P(A' \cap B') = .1$. Equation 1 can then be used to calculate all of the unions, and Equations 3 can be used to calculate all of the conditionals.

Figure 1 illustrates some of the relationships imposed by Equations 3. Probability of the conjunction of A and B should be proportional to P(A), with the slope equal to $P(B|A)$. Specifying two of the values determines the third. The conjunction could also be analyzed as the product of P(B) and $P(A|B)$. The conjunction is therefore constrained by the following four expressions:

$$P(A \cap B) \leq P(A) \quad (5)$$
$$P(A \cap B) \leq P(B) \quad (6)$$
$$P(A \cap B) \leq P(A|B) \quad (7)$$
$$P(A) - P(B') \leq P(A \cap B) \leq P(B|A) \quad (8)$$

which can be seen as consequences of the requirement that all of the values in Equations 3 and 4 must be between 0 and 1.

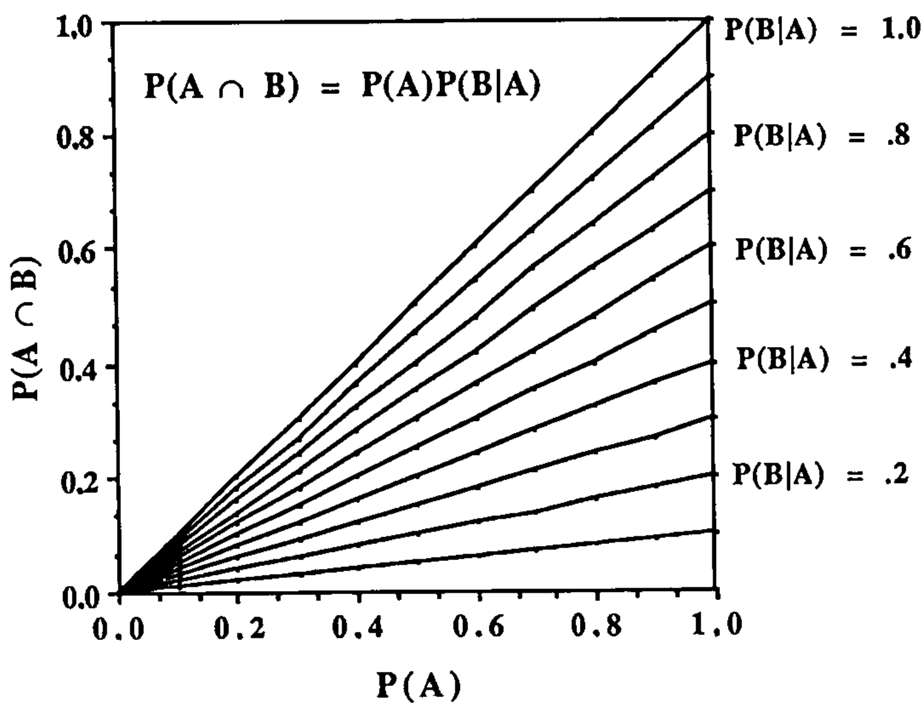

Figure 1. Probability of conjunction of A and B, plotted as a function of the probability of A, with separate curves for different levels of the probability of B given A. Note that $P(A \cap B)$ is always less than or equal to P(A), and it is also less than or equal to $P(B|A)$.

Judgments of probability violate these inequalities. Wyer (1976) found many instances in which the judgment of P(A∩B) was greater than the judgment of either P(A) or P(B). Tversky and Kahneman (1983) reviewed this literature, presented additional examples, and used the term "conjunction fallacy" to describe violations of Expressions 5 and 6 that occur in within-subjects designs. Judged probabilities have systematically violated probability theory in a number of studies involving a variety of tasks. Reviews of this literature from different points of view can be found in Wyer (1974), Kahneman, Slovic, and Tversky (1982), Birnbaum and Mellers (1983), and other chapters in this volume. As yet, no theory has been developed to explain all of the phenomena involving judged probabilities.

Subjective and Judged Probability

Deviations between implications of probability theory and judgments of probability may occur because subjective events do not obey the algebra of sets, because subjective probabilities do not obey the calculus of probability theory, or because probability judgments have not been properly scaled. Figure 2 helps to clarify these theoretical distinctions. In the outline in Figure 2, events and information influence subjective probabilities (e.g., s(A) in Figure 2). These values may have an organization of their own that differs from that of probability theory. These impressions are mapped into overt judgments of probability by judgment functions, which assign numerical responses to the subjective impressions. In addition, errors may enter the system in the subjective stages or in the responses. The presence of random errors cause judgments of probability (q(A) in Figure 2) to violate the constraints in Expressions 5-8, even if the subjective probabilities conformed to probability theory.

The first issue is whether people use the same representation of events as the investigator. For example, the English words, "*and*" and "*or*", which are assumed to refer to conjunctions and unions, have different meanings in different sentences. For instance, in the sentence, "to enter this bar, you must have either a driver's licence *or* a military I.D. showing you are older than 21 years," people correctly understand that both I.D.s would be acceptable. Similarly, people take "*and*" to denote the conjunction in the sentence, "to cash a check here you need a driver's licence *and* a credit card."

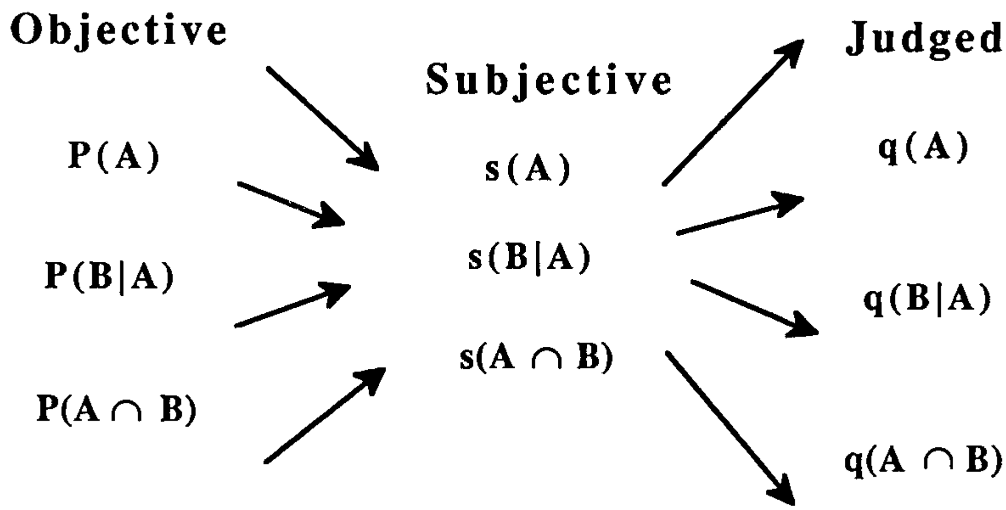

Figure 2. Outline of objective, subjective, and judged probabilities. Objective events, probabilities, and information combine to produce subjective events and their probabilities, which in turn, are mapped into judged probabilities of events.

However, sometimes "*and*" and "*or*" have other meanings. In the sentence, "either you return my wallet *or* I will punch you in the nose," people would be surprised if the wallet were returned and the punch were delivered, presumably because "*or*" in some cases implies the union excluding the intersection, as it does in the sentence, "an assistant professor must either publish *or* perish". The term, "*and*" sometimes refers to the union, as in the sentence, "only women and children may enter the lifeboats." Because the probability of a union often exceeds the probability of the constituents, miscommunication concerning the word "*and*" could explain conjunction fallacies. Tversky and Kahneman (1983) discussed this possibility and concluded that subjects indeed interpreted their conjunction problems in the fashion intended. In this chapter, the issue of subjective algebra of events will also receive less emphasis, but a complete theory of probabilistic reasoning will eventually need to include a theory of the subjective analysis of propositions (Wason & Johnson-Laird, 1972).

The second issue is to determine the calculus of subjective probability. If people do not obey probability theory, what laws would explain probability judgments? Perhaps another coherent structure interlocks judgments of probability in some consistent, if strange, fashion. It is possible that intuitive probability judgments form a "workable" structure that differs from the system in probability theory. Perhaps they form a system that would be considered acceptable for certain situations, but unacceptable for others. For example, people might be fairly accurate at estimating the value of a basket of groceries, but when cash is to be exchanged, both parties feel better when intuitive judgments can replaced by explicit calculations in the supermarket. Nevertheless, investors will buy and sell real estate where the calculations by necessity are much cruder. Similarly, an engineer may have expertise enough to realize whether the design of a freeway overpass will withstand earthquakes of expected magnitudes, but people want to see (and the law requires) explicit calculations, because expert intuitions are sometimes wrong. The point here is that we need to explain the facts that people can be fairly accurate and consistent and yet can still make systematic deviations.

The third issue is to determine the relationship between subjective and judged probability. This issue is analogous to the concept of the judgment function in psychophysical and social judgment (Birnbaum, 1982). Some investigators have implicitly assumed that judged probabilities can be interpreted as subjective probabilities. However, Varey, Mellers, and Birnbaum (in press) observed that judged proportions in visual displays show the same sorts of contextual effects as do other types of psychophysical stimuli. For example, when the actual proportion of open dots in a display of open and solid dots was .17, the judged proportion could be either .21 or .32, depending on the distribution of other proportions presented. Findings such as these indicate that theoreticians need to distinguish between subjective probabilities, which might obey certain consistency properties, and judged probabilities, which might not.

Contextual effects in judgments of proportions indicate that between-subject comparisons of probability judgments need a theory of the context for their interpretation. Birnbaum and Mellers (1983) found that within- and between-subject investigations of the "base rate fallacy" in Bayesian inference problems give very different results. In between-subject studies, the data led some to conclude that subjects underweight the base rate (Kahneman, et. al., 1982). Birnbaum and Mellers (1983) found that when subjects were asked to make a single judgment, in a between-subject design, they appeared to choose one of the stimulus values as their response; however, in within-subject designs, subjects appeared to give weight to the base rate, although they did not combine it with the evidence in the appropriate fashion. Instead of using Bayes Theorem, they appeared to combine the information by a configurally weighted, scale-adjustment averaging model (Birnbaum & Stegner, 1979; 1981). Birnbaum (1982) similarly found that the judged fault of a rape victim who was described as a "virgin" was greater than that of a "divorcee" in a between-subject design, but not when the subjects judged both types of victims or even judged the fault of the defendant in a within-subject design. Birnbaum argued that Parducci's (1965;1983) range-frequency theory could explain these effects if it is postulated that in a between-subject design, the stimulus and the context are confounded, because the stimulus brings its own context. It may be that the subjective "fault" of these victims never changed, only the judgments.

Suppose probability theory were an accurate representation of intuitive reasoning, but that judged probabilities were linear functions of subjective probabilities, with errors that produce regression. This concept is similar to the general idea of factor analysis, in which there are "true" values of psychological constructs, and observed measures are assumed to be correlated with them. Let us suppose also that the marginal distributions of true scores and observed scores (judgments) are constant, consistent with range-frequency theory. If so, then statistical regression will influence the slopes and intercepts of the functions relating average judgments to the "true" scores, and therefore also influence the relationships between different judgments. [In the diagram in Figure 2, notice that errors could be attributed to the psychological processing as well as to the response stages. This distinction is analogous to the distinction made in theories of scaling and signal detection between variability in the perception of the stimuli and variability in the criteria, or limens for judgment (see e.g., Torgerson, 1958; Thurstone, 1927). This distinction allows the subject to have uncertainty about a stimulus (fuzziness in the subjective probability) as well as uncertainty about what to call it.]

Conjunction errors can be explained by simply allowing that the regression of judgments of conjunction probabilities has a lower slope relative to other probability judgments. Taking the picture of Figure 1 into account, the idea of regression suggests a simple recipe for finding violations of Expressions 5 and 6 (conjunction fallacies): combine small values of P(A) with large values of P(B|A) or small values of P(B) with large values of P(A|B). One might expect to find violations of Expressions 7 and 8 when the probability of one event is high and the conditional probability of the other event given that event is low.

EXPERIMENTAL INVESTIGATIONS OF CONJUNCTIONS

Wyer (1976) proposed a type of averaging model for conjunction probabilities. However, Tversky and Kahneman (1983) argued against averaging models on the basis of their results with the "Peter" problem. Instead, they concluded that psychological "representativeness" introduces a bias into judgments of probability. Birnbaum, C. Anderson, and Hynan (1989) designed a study of the "Peter" problem to address three issues: First, they used variations of the "Peter" problem, to check whether or not the "conjunction fallacy" would behave like the "base rate fallacy", and change drastically when the subject makes many judgments. Second, each subject was requested to judge events, conjunctions, and conditionals to allow tests of Expressions 5-8 and also, presumably, to keep the subject clear on the distinctions among these different concepts. Third, structure was imposed on the problems to facilitate tests of models of the conjunction fallacy.

In one of their experiments, Birnbaum, *et al.* (1989) asked 60 university students to evaluate 56 variations of the "Peter" problem (Tversky & Kahneman, 1983, p. 306), which are all based on the following information:

> Peter is a junior in college who is training to run the mile in a regional meet. In his best race, earlier this season, Peter ran the mile in 4:06 min.

The subjects were asked to judge the probabilities of various outcomes. For example, what is the probability that Peter will run the second half-mile under 2:04 min. and will complete the mile under 4:09 min.? They judged the probabilities that Peter will complete the whole Mile in under 4:00, 4:03, 4:06, or 4:09 min. (Event A); they also judged the probabilities for second Half Mile times under 1:55, 1:58, 2:01, or 2:04 min. (Event B). Each event was defined as Peter's time being under the time specified. Subjects also judged all of the 16 (4 by 4) conjunctions of these times, and all 16 conditional probabilities of the Mile times given Half mile times, and also all of the 16 conditional probabilities of Half mile times given whole Mile times. After warm up trials, subjects rated all of these 56 events, conjunctions, and conditionals, randomly ordered (8 Events, A, B; 16 conjunctions, 16 conditionals of A | B and 16 conditionals of B | A).

Table 1 shows the mean judgments of the conjunction probabilities, with the mean judgments of the events in the last row and column. If Expressions 5 and 6 described judgments, then each judged probability of a conjunction should be less than the corresponding judgments for the constituent events. Instead, all of the entries in the first and second rows of conjunctions exceed the judged probability for the second Half mile (last column). Similarly, all of the conjunctions to finish the Mile under 4:00 (first column) with any Half mile exceed the probability of the 4:00 min. Mile (last row of first column). Table 2 shows that mean judgments of conditional probabilities can also be exceeded by conjunctions, contrary to Expressions 7 and 8. For example, all conjunctions involving the 4:09 min Mile exceed the corresponding judgments of conditional probabilities of finishing the Half mile times given the slow, 4:09 time for the Mile (last column of lower portion of Table 2).

Table 1. Mean Judgments of Events and Conjunctions.

Half Mile	Whole Mile				q(B)
	4:00	4:03	4:06	4:09	
B	.29	.33	.38	.37	.22
1:55	.27	.36	.48	.42	.25
1:58	.26	.44	.46	.57	.42
2:01	.29	.38	.57	.67	.65
2:04	.23	.36	.58	.81	
q(A)					

Note: A = Peter finishes the whole Mile under time listed; B = Peter's time for the second Half mile of the same race is less than time listed. From Birnbaum *et al.* (1989).

Two situations are highlighted in Table 3 that appear inconsistent with the account of Tversky and Kahneman (1983). The representativeness interpretation suggests that two fast times (4 min Mile with a 1:55 Second Half) should produce a conjunction fallacy because that combination would be representative of a winning race. However, the 4:09 Mile and fast 1:55 Half mile, though not representative, produce a greater conjunction judgment, and a greater "fallacy" when compared with the Half mile event. As will be shown below, these results are consistent with an algebraic explanation that uses the concept of subjective conditional probability instead of representativeness.

Table 2. Mean Judgments of Conditional Probabilities.

Conditionals: Whole Mile given Half Mile [q(A|B)]				
	A			
B	4:00	4:03	4:06	4:09
1:55	.68	.67	.78	.82
1:58	.60	.62	.76	.78
2:01	.57	.58	.74	.82
2:04	.36	.35	.56	.75
Conditionals: Half Mile given Whole Mile [q(B|A)]				
1:55	.49	.36	.30	.27
1:58	.54	.43	.37	.32
2:01	.60	.60	.53	.40
2:04	.55	.63	.66	.65

Note: see note to Table 1. From Birnbaum *et al.* (1989).

Table 3. Comparison of Probability Judgments for Two Situations.

Judgment	Mile < 4:00 min Half < 1:55 min.	Mile < 4:09 min. Half < 1:55 min.
P(Mile)	.23 (45%)	.81 (7%)
P(Half)	.22 (48%)	.22 (55%)
P(Mile|Half)	.68 (15%)	.82 (2%)
P(Half|Mile)	.49 (13%)	.27 (47%)
P(Mile∩Half)	.29	.37

Note: Percentages of subjects who judged conjunction more probable than each term (violating Expressions 5-8) are shown in parentheses. From Birnbaum, *et al.* (1989).

Table 3 also shows the percentage of subjects whose conjunction judgments were greater than their judgments of the events and conditionals. For example, for the 4:00 min. Mile and 1:55 min. Half mile, the conjunction was judged greater than the probability of the 4 min. Mile by 45% of the subjects and greater than the 1:55 min. Half by 48% of the subjects, in violation of Expressions 5 and 6; 38% judged the conjunction more probable than both of the marginal events. When the Mile time was increased to 4:09 min. with the same Half mile time, 55% of the subjects rated the conjunction more probable than the 1:55 Half mile, 7% judged it more probable than the 4:09 Mile, and only 5% judged it more probable than both marginal events. This conjunction was also judged more probable than the judged conditional probability of finishing the Half under 1:55 given the Mile is under 4:09 by 47% of the subjects, in violation of Expression 8. For the two slowest times (4:09 Mile and 2:04 Half), 25%, 60%, 32%, and 35% of the subjects violated Expressions 5, 6, 7, and 8, respectively; 18% judged the conjunction more probable than both marginal events.

Fit of Probability Theory

The standard probability theory was fit to the data by estimating the values of subjective probabilities for the events and the conditionals to minimize the sum of squared errors in the following set of equations:

$$q(A) = s(A) + e_A \quad (9)$$
$$q(B) = s(B) + e_B \quad (10)$$
$$q(A|B) = s(A|B) + e_{A|B} \quad (11)$$
$$q(B|A) = s(B)s(A|B)/s(A) + e_{B|A} \quad (12)$$
$$q(A \cap B) = s(B)s(A|B) + e_{A \cap B} \quad (13)$$

where $q(A)$, $s(A)$, and e_A denote the judged probability of A, the subjective probability of A, and random error in the judgment of A, respectively. Note that in Equations 9, 10, and 11, there are as many subjective values as judgments, but the same parameters reappear in Equations 12 and 13.

To fit the model to the data, a special computer program was written to select 24 scale values [4 values of s(A), 4 values of s(B), and 16 values of s(A|B)] so as to minimize the sum of squared errors. The program utilized the STEPIT subroutine (Chandler, 1969) to minimize the following index of fit: $F = \Sigma(q_i - q'_i)^2$, where F is the function to be minimized; q_i is the mean judgment; q_i is the predicted judgment; and the summation is over the 56 judgments of probability.

This version of probability theory achieved a rather poor index of fit of .33, using 24 scale values to approximate 56 judgments. Apparently, the equations of probability theory do not fit the judgments well, even when the subjects make a variety of judgments in a within-subjects design, and even when the data are fit in a fashion that allows the errors to occur in all dependent variables. The problem is that this model predicts that conjunction judgments should be less than their constituents, as in Expressions 5-8, but contrary to the data.

When probability theory is modified to allow the dependent variables to be only linearly related to subjective probability, (allowing judgment functions to intervene between subjective probability and response), then the algebra of probability theory gives a much better fit. Two additional parameters can be introduced into Equation 13 as follows:

$$q(A \cap B) = cs(B)s(A \mid B) + d + e_{A \cap B} \qquad (13a)$$

where c and d are constants that allow judgments of conjunctions to be linearly related to the other judgments of probability. This model has an overall sum of squared errors of .06, which is a vast improvement over the unmodified probability theory. This model accounts for the general pattern of violations of the Expressions 5-8.

Wyer (1976) proposed a model that can also be interpreted as probability theory with different scales for the event and conditional probabilities, although he interpreted it as a compromise between the average of the event and conditional probability and the product of these terms. (If the additive constants differ for different scales, then the equation for conjunctions will be a linear combination of the product and the components.) Wyer's model was generalized and fit using the approach described above, and it achieved a fit comparable to that of the modified probability theory.

The conditional probabilities and event probabilities showed a fair degree of consistency with the equations of probability theory. For example, for each subject's data, the relationship between the products, $q(A)q(B \mid A)$ and $q(B)q(A \mid B)$, was examined; interestingly, these two products fell close to the identity line in most cases, as would be predicted from Equation 12. This consistency should be regarded with caution, because if there were distortions in the conditional probability judgments, such tests of consistency might allow biases to go undetected since they appear on both sides of the expression under investigation.

Examination of the data, however, indicated that the conjunctions still showed systematic deviations from the predictions of the modified probability theory that appeared to indicate that conjunction judgments should be represented as a geometric average of s(A) and s(B|A). Furthermore, it seems unattractive to postulate different judgment functions for the different types of probability judgments since they were obtained under the same conditions with the same stimuli.

A General Theory

The following theory was developed to include as special cases both probability theory and geometric averaging theory, while retaining a consistency of the conditional probabilities, analogous to Equations 3. Suppose the subjective analog to Equations 3 can be written as follows:

$$s(A \cap B) = s(A|B)^{\alpha} s(B)^{\beta} = s(B|A)^{\alpha} s(A)^{\beta} \qquad (14)$$

where α and β are the weights in the geometric average determining the intuitive probability of the conjunction. These equations lead to a generalized theory in which the conditional probabilities are proportional to each other, and the proportion is a function of the ratio of event probabilities, as in probability theory, but a power function of the ratio connects the two conditionals, as follows:

$$q(B|A) = s(A|B)[s(B)/s(A)]^{\beta/\alpha} + e_{B|A} \qquad (15)$$

$$q(A \cap B) = s(A)^{\beta} s(B|A)^{\alpha} + e_{A \cap B} \qquad (16)$$

where Equations 15 and 16 replaced Equations 12 and 13, and Equations 9-11 were kept the same. When $\alpha = \beta = 1$, Equations 14 reduce to the subjective counterpart of Equations 3, and Equations 15 and 16 are the same as Equations 12 and 13. When $\alpha = 1 - \beta$, Equation 14 is a geometric averaging model.

The fit of several special cases of this model are shown in Table 4. When $\alpha = \beta$ and $\alpha + \beta = 2$, the theory reduces to probability theory, and the fit is .33. When both parameters were free to vary, the best-fit values were $\alpha = .68$ and $\beta = .54$, and the overall fit was

.05. Examination of the predictions indicated that this version of the model gave a better approximation to the data than the modified probability theory, which used the same number of parameters and allowed a different judgment function for conjunctions.

Table 4. Fit of Special Cases of the Generalized Model.

Constraints on the Parameters		
	$\alpha = \beta$	α, β unrestricted
$\alpha + \beta = 2$	.33	.33
$\alpha + \beta = 1$	.11	.10
$\alpha + \beta$ free	.06	.05

Note: Index of fit is sum of squared deviations between means and model predictions.

MODELS, HEURISTICS, AND BIASES

The algebraic modelling approach does not make use of the heuristic of representativeness, or other content-specific cognitive biases. The approach makes the strong prediction that the content of the problem makes no difference, except to affect the parameters of the model. In using an algebraic model, the present chapter is compatible with the approach of Wyer (1976). In contrast, Tversky and Kahneman (1983) approach conjunction fallacies in terms of the specific content of the problems, citing the biasing effects of "representativeness" on probability judgments (see also Kahneman & Tversky, 1972; 1980; Kahneman, *et al.*, 1982; Slovic, Lichtenstein, & Fischhoff, 1988).

Representativeness is postulated to be a psychological construct that co-exists with subjective probability, that is "naturally assessed", and that contaminates subjective probabilities, which presumably might otherwise be unbiased. Representativeness is analogous to, but distinct from similarity or prototypicality, and it can be measured by asking subjects to rate the "representativeness" of certain events. The theory of representativeness can be depicted as in Figure 3. Events and stimulus information affect two constructs, subjective probabilities and representativeness, which in turn both influence the dependent variables, subjective probability and judged representativeness. A complete theory would specify how the independent variables affect each construct, how the constructs influence each other, and how they affect the two types of judgments.

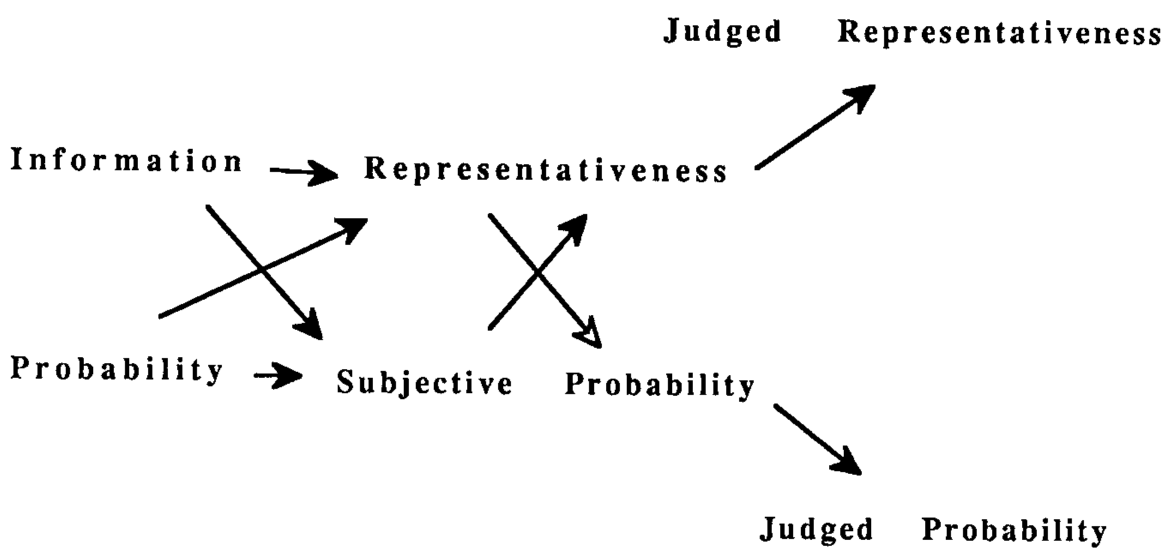

Figure 3. Representativeness as a psychological construct, independent of subjective probability. The open arrow is used to highlight the open question whether or not representativeness has a causal effect on subjective probabilities.

Among the obvious null hypotheses to test are the one-mediator model, which states that both types of judgments are mediated by a single intervening construct, and the two-mediator, segregated theory that there are indeed two mediators of the judgments, but there is no causal effect of representativeness on subjective or judged probability. Single mediator theories have had surprising success in situations in which it had been taken for granted that multiple mediators were at work (Birnbaum, 1982, 1985). For example, even when subjects are instructed to judge "ratios" and "differences" between stimuli, the data can be explained by the theory that both judgments are mediated by the same comparison operation on the same subjective values. Had subjects actually used two different operations to compare the stimuli, the judgments would not have been monotonically related, but instead would have shown a particular ordinal pattern. It should be noted that single mediator theories do not predict that the dependent variables will be perfectly correlated, nor do they require that the partial correlation between an independent variable and one dependent variable with the others partialed out will be zero; instead, these partials are predicted to have the same signs as the original correlations (Birnbaum & Hynan, 1986; Birnbaum, 1985).

The evidence presented does not yet appear to require rejection of such simpler null hypotheses in favor of the theory that the construct of representativeness is distinct enough from subjective probability to be given status as a separate, causal mediator. What has been demonstrated is that judgments of representativeness and probability are correlated, and that both demonstrate certain properties that are not consistent with "standard" theories of probability. We will now consider some examples of conjunction fallacies to contrast heuristic interpretations as opposed to algebraic interpretations in terms of subjective conditional probabilities.

The Linda problem, the Bill problem, and the other experiments cited as examples of the M->A paradigm by Tversky and Kahneman (1983, p. 305) are consistent with the model presented here, assuming plausible values for the parameters. Linda is described as follows:

> "Linda is 31 years old, single, outspoken and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations."

People rank the probability that Linda is a feminist *and* a bank teller ($A \cap B$) higher than the probability that Linda is a bank teller (B). The representativeness interpretation assumes that the description produces a "model" of Linda that resembles a feminist and adding feminist to bank teller makes the conjunction more plausible than leaving it out. The weakness of this explanation is that it "passes the buck" from probability to another psychological construct, representativeness, without explaining why that construct has the properties that it does.

For the present algebraic model, the interpretation would be that the description of Linda produces high values of s(A), low values of s(B), and a high value of $s(A|B)$. For example, suppose s(A) = .85 (she is probably a feminist), s(B) = .1 (she is probably not a bank teller), and $s(A|B)$ = .80 (given Linda is a bank teller, it is still quite likely that she is a feminist). Assuming $\alpha = .68$ and $\beta = .54$ (the values estimated for the Peter problem), Equation 14 implies a value of $s(A \cap B) = .25$, which exceeds P(B). The weakness of this explanation is that the parameters are left to be determined from the data; however, the strength of such an explanation is that it makes particular predictions and can be refuted by evidence, if the experiments are designed to constrain the parameters. The pattern predicted by Equation 14 is shown in Figure 4, plotted for comparison with Figure 1. This figure shows the situations in which judgments of the probability of conjunctions are predicted to exceed judgments of the probability of events or conditionals.

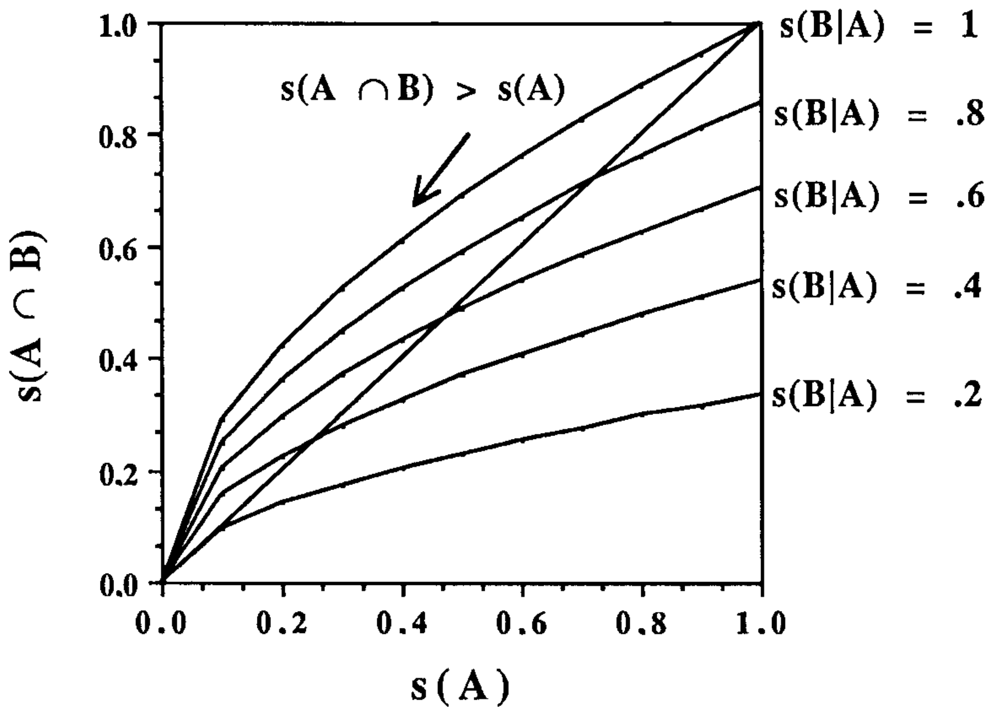

Figure 4. Subjective probability of a conjunction, according to Equation 14, using parameters estimates from Birnbaum, *et al.* (1989). Values are plotted to permit comparison with probability theory, as in Figure 1. Curves above the identity line represent values for which subjective probability of a conjunction exceeds the event probability listed on the abscissa.

Tversky and Kahneman (1983) cited the "Peter" Problem as evidence against an averaging rule. Their argument was based on the finding that many subjects judged the conjunction as more probable than either constituent. However, such a result would disprove only the simple average of the event probabilities, and it would be consistent with any averaging model with an initial impression, even with independent events (Yates & Carlson, 1986). Furthermore, the result is consistent with the generalized averaging model of the event and conditional probability, as in Equation 14. In general, the concept of conditional probability may permit a more tractable theoretical construct and perhaps more testable theory than the construct of representativeness.

The "Peter" problem, the health survey problem, and the other problems cited by Tversky and Kahneman (1983) as examples of the A -> B paradigm can also be explained by the present model. According to the representativeness argument, event A provides a possible cause, explanation, or motive for B and therefore, it makes the conjunction more plausible. By the present model, the link between A and B is represented by subjective conditional probabilities. In these problems, the values of s(A) and s(B) might both be small, but the values of s(A|B) and s(B|A) are high. The Peter problem, with 4 min. Mile and 1:55 min. Half, is an example of this situation. As shown in Table 3, the judged probabilities of these events are low (.23 and .22); however, the conditional probabilities of finishing the Mile in less than 4:00 min., *given* Peter finishes the second Half under 1:55 min., is high (.68). The judged probability that Peter will finish the Half mile under 1:55, *given* a sub-4:00 min. Mile, is also high (.49). Figure 4 shows that the predicted value of the conjunction is higher in this situation than both of subjective probabilities of the separate events.

In the health survey problem, Mr. F. was selected randomly from the participants in a survey of adult males in a certain population. Subjects ranked the probability that Mr. F has had a heart attack *and* is over 55 years old ($A \cap B$) higher than the probability that Mr. F has had a heart attack (A). However, subjects tended to rank the conjunction that Mr. F. has had a heart attack and Mr. G. (another randomly selected participant) is over 55 years old ($A \cap C$) as lower than the probability that Mr. F. has had a heart attack. These two data are both consistent with the present model. Suppose the subjective probability of being over 55 years old is .4 [s(B) = s(C) = .4], and the probability of having a heart attack is .3 [s(A)]. Suppose the probability that Mr. F is over 55, given he has had a heart attack is .8 [s(B|A) = .8]. It seems reasonable to suppose that the probability that Mr. G. is over 55, given Mr. F has had a heart attack is still .4 [s(C|A) = s(C)]. Hence, the two conjunctions are predicted to be s($A \cap B$) = .45, which exceeds s(A), and

$s(A \cap C) = .28$, which is (properly) less than both s(A) and s(B). These situations are illustrated in Figure 4, where it can be seen that $s(A \cap B)$ can be greater than or less than s(A), depending on the value of $s(B|A)$. In each of these cases, the concept of subjective conditional probability seems a natural replacement for the concept of representativeness. Furthermore, because it can be quantified and specified in a model, it allows an explicit prediction of when the conjunction will violate the inequalities imposed by probability theory.

If a concept is not required to deduce the phenomena to be explained, by Ockham's razor, the construct can be eliminated from the theory of the phenomena without loss. It will be useful to devise experiments in which the construct of representativeness is required and is also testable. The theory of representativeness appears to be one that can be stated in a form that can be tested, but it remains as yet untested.

Yates and Carlson (1986) elaborated the theory of Tversky and Kahneman (1983) that subjects make "natural assessments" of similarity or representativeness; they postulated that there could be a variety of strategies or procedures that might be evoked by different situations. To test this hypothesis would require demonstrating that some variable (such as a change in instructions or "set") produced a change that could not be explained by a change in parameters of an unchanging algebraic rule, but instead required a change in algebra. There are subtle changes in wording that seem likely to induce the subject to utilize different rules. For example, consider the following problems:

1. If you were taking an examination, which of the following choices would most probably be marked as correct?
   a. men tend to be taller than women.
   b. men tend to be heavier than women.
   c. both a and b.

2. In a survey of adults, M. and F., a male and female, were selected at random; which statement is most probably correct? Which statement would you bet on?
   a. M. is taller than F.
   b. M. is heavier than F.
   c. M. is both taller and heavier than F.

According to the conventions of testing alluded to in the first problem, when both choices are separately "true", then the choice of "both" is considered correct. However, in the second form of the problem, c is the least probable. If the answers to different forms of a single problem produced different solutions, it would not necessarily imply that the subject's rule changed, since a change in response could be produced by different values of the parameters in an unchanging algebra. Therefore, to demonstrate the locus of the effect of the instructional "set", it would be necessary to conduct a complete investigation that would allow one to estimate the parameters and test the algebraic model of each type of problem. It remains to be demonstrated that changes in the problem produce changes in the strategies used by subjects.

Such investigations may reveal how subjective and judged probabilities are formed in different contexts. There may exist a set of procedures that would allow an investigator to elicit judgments of probabilities that would obey a consistent algebra, so that the subjective probabilities, defined as parameters in the model, might have some use in guiding decisions. If subjective probabilities are to determine the future, we have a right to know what their properties are, how they can be measured, and how they can be properly employed in making better decisions.

REFERENCES

Birnbaum, M.H. (1982). Controversies in psychological measurement. In B. Wegener (Ed.), Social attitudes and psychophysical measurement. Hillsdale, NJ: Erlbaum, 401-485.

Birnbaum, M.H. (1985). Relationships among models of salary bias. American Psychologist, 40, 862-866.

Birnbaum, M.H., Anderson, C.J., & Hynan, L.G. (1989). Conditional Probability and the Conjunction Fallacy. Unpublished manuscript. Available from author, Dept. of Psychology, C.S.U.F., Fullerton, CA 92634.

Birnbaum, M.H., & Hynan, L.G. (1986). Judgments of salary bias and test bias from statistical evidence. Organizational Behavior and Human Decision Processes, 37, 266-278.

Birnbaum, M.H., & Mellers, B.A. (1983). Bayesian inference: Combining base rates with reports of sources. Journal of Personality and Social Psychology, 45, 792-804.

Birnbaum, M.H., & Stegner, S.E. (1979). Source credibility in social judgment: Bias, expertise, and the judge's point of view. Journal of Personality and Social Psychology, 37, 48-74.

Birnbaum, M.H., & Stegner, S.E. (1981). Measuring the importance of cues in judgment of individuals: Subjective theories of IQ as a function of heredity and environment. Journal of Experimental and Social Psychology, 17, 159-182.

Chandler, J.P. (1969). STEPIT: Finds local minima of a smooth function of several parameters. Behavioral Science, 14, 81-82.

Hogg, R.V., & Craig, A.T. (1965). Introduction to mathematical statistics. New York: Macmillan.

Kahneman, D., & Tversky, A. (1972). Subjective probability: A judgment of representativeness. Cognitive Psychology, 3, 430-454.

Kahneman, D., & Tversky, A. (1980). Causal schemas in judgments under uncertainty. In M. Fishbein (Ed.), Progress in social psychology. Hillsdale, NJ: Erlbaum.

Kahneman, D., Slovic, P., & Tversky, A. (1982). Judgment under uncertainty: Heuristics and biases. New York: Cambridge University Press.

Krantz, D.H., Luce, R.D., Suppes, P., & Tversky, A. (1971). Foundations of measurement (Vol. 1). New York: Academic Press.

Kyburg, H.E., & Smokler, H.E. (1964). (Eds.) Studies in subjective probability. New York: Wiley.

Mosteller, F., Rourke, R.E.K., & Thomas, G.B. (1961). Probability and statistics. Reading, MA: Addison-Wesley.

Parducci, A. (1965). Category judgment: A range-frequency model. Psychological Review, 72, 408-418.

Parducci, A. (1983). Category ratings and the relational character of judgment. In H.-G. Geissler (Ed.) Modern issues in perception. Berlin: VEB Deutscher Verlag der Wissenschaften. p. 262-280.

Savage, L.J. (1954). The foundations of statistics. New York: Wiley.

Slovic, P., Lichtenstein, S., & Fischhoff, B. (1988). Decision making. In R.C. Atkinson, R.J. Herrnstein, G. Lindzey, & R.D. Luce (Eds.), Stevens' handbook of experimental psychology (Vol 2). New York: Wiley.

Thurstone, L. L. (1927). A law of comparative judgment. Psychological Review, 34, 273-286.

Torgerson, W. (1958). Psychological scaling. New York: Wiley.

Tversky, A., & Kahneman, D. (1983). Extensional versus intuitive reasoning: The conjunction fallacy in probability judgment. Psychological Review, 90, 293-315.

von Winterfeldt, D. & Edwards, W. (1986). Decision analysis and behavioral research. Cambridge: Cambridge University Press.

Varey, C., Mellers, B.A., & Birnbaum, M.H. (in press). Judgments of proportions. Journal of Experimental Psychology: Human Perception and Performance.

Wason, P.C., & Johnson-Laird, P.N. (1972). Psychology of reasoning. Cambridge, MA: Harvard University Press.

Wyer, R.S. (1974). Cognitive organization and change: An information-processing approach. Hillsdale, NJ: Lawrence Erlbaum.

Wyer, R.S. (1976). An investigation of the relations among probability estimates. Organizational Behavior and Human Performance, 15, 1-18.

Yates, J.F., & Carlson, B.W. (1986). Conjunction errors: Evidence for multiple judgment procedures, including "signed summation." Organizational Behavior and Human Decision Processes, 37, 230-253.