Chapter 8
Behavioral Models of Decision Making Under Risk

Michael H. Birnbaum

Abstract This chapter reviews experiments testing theories of how people make choices between risky prospects, gambles in which the consequences and their probabilities are specified. When people prefer a small amount of cash to the expected value of a gamble, they are said to be risk averse. The St. Petersburg paradox is an extreme case of risk aversion in which people prefer a small cash payment rather than one chance to play a gamble of infinite expected value. Expected utility theory was proposed to explain this paradox by allowing that the utility of money is not a linear function of its cash value but instead shows diminishing marginal returns. Allais proposed two paradoxes that contradicted expected utility theory, and a number of modern theories have been proposed to explain the Allais paradoxes. Among these are original and cumulative prospect theory, configural weighting theory, the priority heuristic, and others. The chapter notes that some decisions are based on experience, where consequences and their probabilities are learned. The chapter also considers models of the variability in decision behavior. New critical tests and their results are reviewed that conclude that neither version of prospect theory can be retained as accurate descriptions of choice behavior and that tests of the heuristic models have yielded data that systematically violate the predictions of those models. The configural weight models remain the best description of the evidence so far accumulated.

Some decisions are based on vague ideas or beliefs of the exact consequences of one's actions given imprecise, uncertain, or ambiguous information concerning the probabilities of consequences contingent on one's alternative courses of actions. For whom should I vote? What job should I take? Should I marry this person? Should I undergo the medical operation my doctor recommended? Such decisions are made in the face of *uncertainty*. The term, *decision making under risk*, in contrast, refers to situations in which a decision-maker has valid information concerning the exact consequences and the probabilities of consequences of the alternative courses of action. For example, should I buy a lottery

M. H. Birnbaum (✉)
Decision Research Center, California State University, Fullerton, CA, USA
e-mail: m@fullerton.edu

M. Raue et al. (eds.), *Psychological Perspectives on Risk and Risk Analysis*,
https://doi.org/10.1007/978-3-319-92478-6_8

ticket for $1 that has one chance in a million of paying $1 million dollars? Researchers studying decisions under risk are attracted to such questions because gambles defined on events with known probabilities (such as tosses of fair coins or rolls of dice) allow one to manipulate important ingredients in the decision process itself, separated from the mechanisms by which beliefs about probability are formed.

Behavioral models of risky decision making are theories that attempt to give empirically accurate descriptions of what people do when confronted with risky decision making problems. Whereas a normative model specifies what a person ought to do to stay consistent with certain principles of rationality, a behavioral model seeks to explain the empirical choices that people actually make, whether these actual choices are deemed rational or not.

In the simplest paradigm for study of decision making, researchers ask participants to make decisions among gambles stated in terms of probabilities to receive monetary consequences. For example, would you rather have $45 for sure, or would you prefer instead to play a risky gamble in which you have a 50–50 chance to win $100 or $0, based on the toss of a fair coin? Because the coin has a probability of ½ to be called correctly, you have a probability of ½ to win $100 and a probability of ½ to win $0. Most people prefer $45 for sure to the risky gamble, even though the gamble would pay $50 on average. This systematic preference for the sure thing contradicts a rule called *expected value*, which was once thought to be a rational principle a person should follow.

Expected Value

The expected value (EV) of a gamble is the mean value of the consequences of a gamble, weighted by their probabilities. Suppose a random process has n possible mutually exclusive and exhaustive outcomes, and let gamble $G = (x_1, p_1; x_2, p_2; x_3, p_3; \ldots; x_i, p_i; \ldots; x_n, p_n)$ represent a gamble with probability p_i to receive consequence x_i, where x_i is the monetary consequence if outcome i occurs. Because the outcomes are mutually exclusive and exhaustive, $\sum p_i = 1$. We can define the EV of gamble G as follows:

$$\mathrm{EV}(G) = \sum p_i\, x_i$$

The gamble based on a coin flip is denoted G = ($100, ½; $0, ½), and G has an EV of $50 = (0.5)($100) + (0.5)($0). So, on average, a person who could play G infinitely many times would expect to win $50 per play, but on any single play of the gamble, the person would win either $0 or $100.

Risk Aversion and St. Petersburg Paradox

EV seemed a reasonable objective measure to many scholars in the eighteenth century, so they considered it paradoxical that when given a choice, people did not always prefer the option with the higher EV. For example, many people prefer \$45 for sure to G = (\$100, ½; \$0, ½), even though the sure thing has a lower EV (\$45) than the gamble (\$50). When people prefer a sure thing to a gamble with the same or higher EV, they are said to be "risk averse."

Risk aversion seemed puzzling, especially when scholars realized that one could construct gambles with infinite value, and people preferred quite small amounts of cash to such gambles. For example, suppose we toss a coin, and if it is heads, you win \$2, but if it is tails, we toss again. On the next toss, if it is heads, the payoff is \$4, and if tails, we toss again. Each time that tails occurs, the prize for heads on the next toss doubles. The expected value of this gamble is as follows:

$$\mathrm{EV} = \$2(1/2) + \$4(1/4) + \$8(1/8) + \$16(1/16) + \ldots = \infty$$

If a person conformed to EV, she should prefer this gamble to any finite amount of money one might offer; indeed, a person should prefer playing this gamble once to all of the money in the world. Yet, most people say they would choose \$20 for sure to one chance to play this gamble, even though the gamble has infinite expected value.

This preference for the sure thing over such gambles (with infinite EV) is now called the *St. Petersburg paradox*, which was discussed in a classic paper by Bernoulli (1738/1954), who presented his paper in St. Petersburg. Bernoulli said it was not necessarily rational to follow EV, but instead to choose the option with the best expected utility.

Expected Utility

Bernoulli (1738/1954) provided a theory of risk aversion that addressed the original versions of the St. Petersburg paradox. This theory proposed that the utility of money is not necessarily equal to its objective value, but might, instead, be a nonlinear function of money. Let $u(x)$ represent the utility (subjective value) of a certain amount of wealth, x. Define expected utility (EU) as follows:

$$\mathrm{EU}(G) = \sum p_i\, u(x_i) \tag{8.1}$$

where $u(x_i)$ is the utility of objective value x_i.

Bernoulli theorized that utility of money might be a logarithmic function of wealth, but he acknowledged that other functions, such as the square root function, might also work. Both of these functions are negatively accelerated; that is, there is

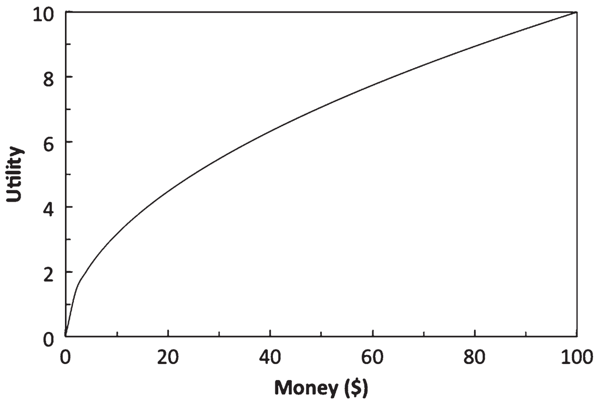

Fig. 8.1 Negatively accelerated utility function. In this example, the value of x whose utility is halfway between \$0 and \$100 is \$25, because $u(x) = x^{0.5}$

a diminishing marginal increment in utility of each additional dollar to the overall utility of wealth; that is, $u(W + x) - u(W)$ decreases as W (wealth) increases for a given increment in wealth, x. For example, if a person had a total wealth of $W = \$50$, an increase of $x = \$100$ in wealth would have a much greater impact than if the person had a total wealth of \$1 million. Figure 8.1 illustrates a hypothetical utility function: $u(x) = x^{0.5}$ (a power function with exponent of 0.5 is also known as the square root function). In this negatively accelerated function (Fig. 8.1), the subjective increase in utility from \$0 to \$25 is the same as the subjective increase in utility from \$25 to \$100. Thus, it takes a bigger increase of money as one moves up the wealth scale to produce the same increase in utility.

Expected utility theory is the theory that people prefer A over B if and only if $\text{EU}(A) > \text{EU}(B)$; that is, A will be preferred to B whenever the expected utility of option A exceeds that of B. EU theory could explain not only risk aversion and the original St. Petersburg paradox, but it could also explain why a pauper who was given a lottery ticket should be happy to sell it for less than EV and why a rich person should be happy to buy it at the same price.

For example, imagine a pauper whose total wealth is just \$50 and who is given a choice between $S = \$45$ for sure and $G = 50–50$ gamble to win \$100 or \$0. Suppose $u(x) = x^{0.5}$, as in Fig. 8.1. EU theory then implies that the utility of choosing the sure thing, S, is $u(\$50 + \$45) = u(\$95) = 9.75$. In contrast, the EU of choosing gamble G is $u(\$50 + \$100)(0.5) + u(\$50) = 9.66$. Because $\text{EU}(S) > \text{EU}(G)$, the theory says the pauper would prefer \$45 for sure over gamble G. Thus, the pauper who was given a lottery ticket (a chance to play gamble G) would be happy to sell it for \$45.

Now consider a richer person whose total wealth is \$1000, who is deciding whether to buy gamble G from the pauper. The utility of Q, the status quo (to not buy) is $u(\$1000) = 31.62$, assuming again that $u(x) = x^{0.5}$. The utility of B, the option to buy the gamble for \$45 from the pauper, has expected utility of $u(\$1000 + \$100 - \$45)(0.5) + u(\$1000 - \$45)(0.5) = 31.69$. Because $\text{EU}(B) > \text{EU}(Q)$, this person should prefer to buy the gamble for \$45. This example shows that even if both people have the same utility function (but different levels of wealth), they can both improve their individual utilities by trading.

It is also reasonable that some people have different utility functions from others, reflecting different attitudes toward risk. For example, if a venturesome person had $u(x) = x^2$, then that person would prefer the risky gamble to a sure thing with the same EV and would be called "risk-seeking." Such a risk-seeking person would even be willing to buy this gamble at a price exceeding \$50 and would outbid the wealthy but risk averse person to buy gamble *G*.

Von Neumann and Morgenstern (1947) showed that expected utility theory could be deduced from four basic axioms of preference and proved that if these axioms are assumed, utility could in principle be measured on an interval scale. The four axioms are *completeness* (for any two lotteries, *A* and *B*, a person either prefers *A* to *B*, *B* to *A*, or is indifferent), *transitivity* [for any three lotteries, *A*, *B*, and *C*, if a person prefers *A* to *B* and *B* to *C*, then the person prefers *A* to *C*], *independence* [for any three lotteries *A*, *B*, and *C*, where *A* is preferred to *B*, and for any probability between 0 and 1, $pA + (1 - p)C$ is preferred to $pB + (1 - p)C$], and *continuity* [for any three lotteries, such that *A* preferred to *B* preferred to *C*, there exists a probability *p* such that *B* is indifferent to $pA + (1 - p)C$]. This axiomatic theory was accepted by many as the definition of what a rational person should do when confronted with decisions under risk.

Much of economic theory had been deduced from the assumptions that people are rational but may differ in their utilities or tastes and that EU theory was rational. For a time, it was also believed that people behave according to this rational theory; therefore, it was thought that classic economic theory not only prescribed what a rational economic actor should do but was also descriptive of actual behavior of individuals. However, both the assumption of rationality of EU and the assumption that people are rational came into question when Allais proposed his paradoxes.

Allais Paradoxes

Allais (1953) criticized EU theory from both descriptive and normative perspectives. He developed paradoxes that have generated continued discussion in the scientific literature that continue to this day, because they revealed contradictions between what seemingly rational people did and what EU theory requires. For example, consider the following two choice problems:

Problem 1: *A*: (\$1 million, 0.11; \$0, 0.89)

B: (\$2 million, 0.10; \$0, 0.90)

Problem 2: *C:* (\$1 million, with certainty)

D: (\$2 million, 0.10; \$1 million, 0.89; \$0, 0.01)

According to EU theory, a person should prefer *C* over *D* if and only if she prefers *A* over *B*; however, many people prefer *C* over *D* and *B* over *A*, contrary to the theory. This paradox is known as the "constant consequence" paradox because 0.89 probability to win \$0 is common to both *A* and *B*, whereas this common consequence of \$0 was changed to a common value of \$1 million in *C* and *D* (in *B*, the common consequence of 0.89 to win \$0 is included in the branch of 0.90 to win \$0). To understand why these preferences violate EU, note that EU(*C*) preferred to EU(*D*) means $u(1\text{ M}) > 0.10u(2\text{ M}) + 0.89u(1\text{ M}) + 0.01u(0)$, which is the same as $0.11u(1\text{ M}) > 0.10u(2\text{ M}) + 0.01u(0)$. However, if EU(*B*) is preferred to EU(*A*), it means $0.11u(1\text{ M}) + 0.89u(0) < 0.10u(2\text{ M}) + 0.90u(0)$, which leads to the contradiction that $0.11u(1\text{ M}) < 0.10u(2\text{ M}) + 0.01u(0)$. If a theory leads to contradiction, it cannot be true.

A "constant ratio" paradox was also developed, which can be illustrated by the following choices:

Problem 3: *E*: \$3000 for sure

F: (\$4000, 0.8; \$0, 0.2)

Problem 4: *G*: (\$3000, 0.25; \$0, 0.75)

H: (\$4000, 0.20; \$0, 0.80)

According to EU theory, a person should prefer *E* to *F* if and only if she prefers *G* to *H*; however, many people prefer *E* to *F* and prefer *H* to *G*. The "constant ratio" refers to the fact that the probabilities to win in *G* and *H* of Problem 4 are a constant ratio (one fourth) of those in *E* and *F* of Problem 3. These paradoxes refuted EU as a descriptive model of how people choose between risky gambles. In the views of Allais (1979), these paradoxes reflected shortcomings of EU as a rational model as well.

Subjectively Weighted Utility and Prospect Theory

Edwards (1954) used a subjectively weighted utility model to account for the Allais paradoxes. According to the model of Edwards, the value of a gamble is given by the following:

$$\mathrm{PV}(G) = \sum w\left(p_i\right)u\left(x_i\right) \tag{8.2}$$

where PV(*G*) is the prospect value of a gamble and $w(p_i)$ is the weight of the probability. Whereas EU theory allowed a nonlinear transformation between objective wealth and utility, this new theory theorized in addition to a nonlinear transformation between objective probability and the (subjective) decision weight assigned to that probability. An example of an inverse S-shaped probability weighting function is shown in Fig. 8.2. In such a function, the weight given to small probabilities is

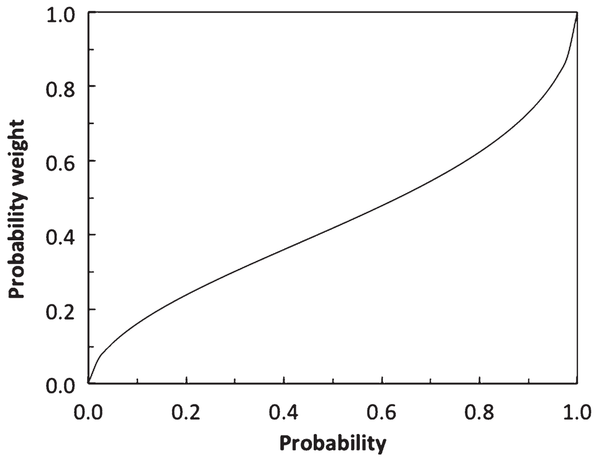

Fig. 8.2 An inverse S-shaped probability weighting function. Note that small probabilities receive weights greater than their probabilities

relatively greater than the objective probability value, and the weight given to large probabilities is lower than the objective probability.

If people placed greater relative weight on small probabilities, as in Fig. 8.2, it could explain why a person who is otherwise risk averse (e.g., for 50–50 gambles) might be willing to buy a lottery ticket that provides only a tiny chance to win a large prize. Edwards also incorporated another revision of EU theory that had been proposed by Markowitz (1952). The utility function in Eq. (8.2) was defined in terms of changes from a reference level rather than absolute wealth. With this revision, x might be either a gain or a loss relative to the status quo, and Edwards (1962) further theorized that different functions might be required for gambles composed strictly of gains, strictly of losses, or combinations of gains and losses.

Tversky, a former student of Edwards, and Kahneman published a variant of this model in *Econometrica* under the name "prospect theory" (Kahneman & Tversky, 1979). As Kahneman (2003) later noted, there was not much new in this paper compared to the literature in psychology, but the paper had a tremendous impact in the field of economics, where it helped inspire the field of behavioral economics, the study of how people actually behave in experiments on economics.

Although "prospect theory" could account for the Allais paradoxes, it made some strange predictions that seemed unrealistic. For example, it predicts that people should prefer gamble I = ($100, 0.01; $99, 0.01; $98, 0.098) to J = ($102; 0.5; $101, 0.5), even though every outcome of J is better than any outcome of I. If the weighting function is nonlinear, and if small probabilities get greater weight, then splitting a certain amount of probability into smaller pieces could increase weight enough to make worse gambles seem better. Because it seemed unlikely that people would violate stochastic dominance (e.g., choose I) in such cases, Kahneman and Tversky (1979) postulated "editing rules" that people supposedly used to avoid such implications of this model, and they postulated other restrictions and exceptions to

Eq. (8.2). Edwards (1954) model (Eq. 8.2) is now sometimes called "stripped" prospect theory, when it is applied without the restrictions and editing rules that were added to prospect theory by Kahneman and Tversky (1979).

Rank-dependent weighting was proposed (Quiggin, 1985, 1993) as a way to account for the Allais paradoxes without violating stochastic dominance. Luce and Fishburn (1991, 1995) developed a generalized version called rank- and sign-dependent utility (RSDU) that allowed different rank-dependent weighting functions for gains and losses (Luce, 2000). Tversky and Kahneman (1992) adopted a version of this model and called it cumulative prospect theory (CPT). According to RSDU or CPT, the value of a gamble on strictly nonnegative consequences is given by the following:

$$\mathrm{CPV}(G) = \sum \left[W\left(P_i\right) - W\left(Q_i\right) \right] u\left(x_i\right) \tag{8.3}$$

where W is a strictly monotonic function from $W(0) = 0$ to $W(1) = 1$ that assigns decumulative weight to decumulative probability, P_i is the decumulative probability to win x_i or more, and Q_i is the probability to win strictly more than x_i.

This CPT model (Eq. 8.3) always satisfies stochastic dominance, and it also satisfies other principles that had required editing rules in original prospect theory. CPT could account for the Allais paradoxes by means of an inverse S-shaped decumulative weighting function, like that in Fig. 8.2 (except that the x-axis now represents decumulative probability). This function assigns more weight to branches leading to smallest and largest consequences of a gamble than to branches leading to intermediate ones.

For a time, CPT appeared a better description than EU theory, but it had not been tested against an earlier approach called "configural weighting" that had been proposed in the 1970s that shared some features of rank-dependent weighting but which differed in important ways.

Configural Weighting Models

Birnbaum (1974; Birnbaum & Stegner, 1979) proposed configural weight models in which the rank of a stimulus affects its weight. Those aspects of a stimulus that are more unfavorable often seem to receive greater weight—a person who is described as "phony and understanding" or "sincere and mean" is not rated as neutral in likeableness, but instead is given a low rating, closer in value to the lower-evaluated information than to the higher. Similarly, a 50–50 gamble to receive either \$0 or \$100 is evaluated closer in value to \$0 than to \$100, as if \$0 gets a greater weight than \$100. Although these rank-affected, configural weight models had much in common with the models later developed independently as "rank-dependent utility," the configural weight models can be distinguished from rank-dependent ones because they make different predictions in certain cases; for example, they do not always imply stochastic dominance.

Configural weighting provides a different interpretation of risk aversion than found in EU theory: according to EU theory, risk aversion is produced by curvature of the utility function (Fig. 8.1); according to configural weighting theory, however, risk aversion or risk-seeking is mainly produced by over- or under-weighting of the lower-valued consequences or aspects of a gamble or stimulus. For example, suppose the utility function is linear, $u(x) = x$, and people give twice as much weight to the lower consequence in a 50–50 gamble as to the higher one. Then the value of a 50–50 gamble to win \$100 or \$0 is \$33. The intuition is that people give extra attention to lower-valued consequences, leaving less weight for higher-valued consequences. A configural weight model that captures this intuition is the transfer of attention exchange (TAX) model, which postulates that attention is diverted ("taxed") from higher-valued outcomes and transferred to lower-valued consequences.

Consider the simple case of a 50–50 gamble to win either x or y, where $x > y \geq 0$. The TAX model for this gamble can be written as follows:

$$\mathrm{TAX}(G) = (0.5 + \omega)u(x) + (0.5 - \omega)u(y)$$

where ω is the configural weight transferred from the lower-valued to the higher-valued consequence or aspect of the gamble or stimulus ($-0.5 \leq \omega \leq 0.5$). If $\omega = 0$, TAX reduces to EU; if $\omega = 0.5$, it becomes a maximum model, and with $\omega = -0.5$, it becomes a minimum model. For gambles on small amounts of cash ($x <$ \$150), with college students, one can approximate $u(x) = x$, and $\omega = -1/6$, so the lowest consequence would get a weight of 2/3 and the highest a weight of 1/3. If a person had these parameters, that person would prefer \$45 for sure to the 50–50 gamble to win \$100, would prefer the gamble to \$20, and would be indifferent between the gamble and \$33 for sure.

For gambles with two possible consequences of the form, $G = (x, p; y)$, $x > y \geq 0$, the TAX model can be written as follows:

$$\mathrm{TAX}(G) = \left[au(x) + bu(y)\right] / (a + b)$$

where a and b are the weights of the higher and lower consequences, which have utilities of $u(x)$ and $u(y)$, respectively. For a risk-averse person, weights in a Special Case TAX model are given as follows:

$$a = t(p) - \delta t(p)/3$$

$$b = t(1 - p) + \delta t(p)/3$$

where $t(p)$ is a function of p, usually approximated as a power function, and $\delta > 0$ is a constant reflecting the transfer of weight (attention) from the higher-valued consequence to the lower-valued consequence. When the transfer goes the other direction, $\delta < 0$, one replaces $t(p)$ with $t(1 - p)$ in the above equations.[1]

With three-branch gambles of the form, $G = (x, p; y, q; z, 1 - p - q)$, $x > y > z \geq 0$, the model is again a weighted average, TAX(G) = $[Au(x) + Bu(y) + Cu(z)]/(A + B + C)$, where the weights (for branches with highest, middle, and lowest consequences) are as follows (in the Special TAX model) for a person who places greater weight on lower-valued consequences:

$$A = t(p) - 2\delta t(p)/4$$

$$B = t(q) + \delta t(p)/4 - \delta t(q)/4$$

$$C = t(1 - p - q) + \delta t(p)/4 + \delta t(q)/4$$

Previous research has shown that modal choices by undergraduates for gambles involving small positive values can be roughly approximated by $t(p) = p^{0.7}$, $u(x) = x$, and $\delta = 1$. Although these "prior" parameters (which were not "best-fit" but were roughly based on previous data) have done fairly well in predicting new group data for the last 20 years, data fitting also shows that the estimated utility function should be negatively accelerated, especially when consequences cover a large range of values.

There are two aspects of the weights that deserve emphasis: First, the transfer of weights has the implication that risk aversion or risk-seeking can be explained by greater or reduced weight on the lower-valued consequence.

Second, the weighting of branches need not satisfy the property of coalescing, which is the assumption that splitting a branch of a gamble would not affect its utility. For example, coalescing implies that A = (\$96, 0.85; \$96, 0.05; \$12, 0.10) should have the same utility as B = (\$98, 0.90; \$12, 0.10). Note that A and B are (objectively) the same; B is called the *coalesced* form of the gamble, and A is one of many possible *split* forms of the same gamble. Instead, splitting a branch in this model increases the weight given to the consequences of the split branch. This implication follows from the fact that $t(p)$ is negatively accelerated, like many other psychophysical functions. In this averaging model, splitting the branch leading to the highest consequences tends to make a gamble A better than B (subjectively). Splitting the branch leading to the lowest consequence would tend to make a gamble seem worse.

Differences in the properties and predictions between the configural weight models and RSDU models including CPT were identified and tested by Birnbaum

[1] As noted in Birnbaum (2008b, p. 471), the convention for ranking consequences was changed from lowest to highest, used in early papers on configural weighting, to highest to lowest, to agree with the conventions used in CPT; therefore, $\delta < 0$ in Birnbaum & Navarrete (1998) corresponds to $\delta > 0$ here and in papers after 2008.

in a series of experiments that refuted this class of models as descriptive of decision making (Birnbaum, 2004a, 2004b, 2006; Marley & Luce, 2005). The configural weight models, based on previous data, correctly predicted where to find new violations, which Birnbaum (2008b) called "new paradoxes" because these critical properties refuted CPT in the same way that Allais paradoxes refuted EU; that is, they lead to contradictions in the model that cannot be explained by revising parameters or functions in the model. More than a dozen critical tests have been devised that reveal that CPT is systematically violated (reviewed in Birnbaum, 2008b, 2008c). Two of these critical tests among these models are reviewed in the next sections.

Violations of Stochastic Dominance

If the probability to win a prize of x or greater in gamble F is always at least as high and sometimes higher than the corresponding probability in gamble G, we say that gamble F dominates gamble G by first-order stochastic dominance. According to rank- and sign-dependent utility theories, including CPT and EU, first-order stochastic dominance must be satisfied. The configural weight models, however, imply that special choice problems can be constructed in which people will violate stochastic dominance.

Birnbaum and Navarrete (1998) tested choice problems such as the following that were predicted by configural weight models (such as TAX) to violate stochastic dominance:

Problem 5: *K*: ($96; 0.90; $14, 0.05; $12, 0.05)

L: ($96, 0.85; $90, 0.05; $12, 0.10)

Birnbaum and Navarrete (1998) found that about 70% of undergraduates choose *L* over *K*, even though *K* dominates *L*. Note that the probability to win $96 is higher in *K* than *L*, the probability to win $90 or more is the same, the probability to win $14 or more is higher in *K* than *L,* and the probability to win $12 or more is the same. There have now been dozens of studies reporting similar, substantial violations of stochastic dominance in choice problems of this type, using different types of participants, different types of monetary incentives, different types of probability mechanisms, different formats for presenting choice problems, and different types of event framing (Birnbaum, 2004a, 2004b, 2006, 2007, 2008b; Birnbaum & Bahra, 2012a). These robust violations indicate that no form of rank- and sign-dependent utility function, including CPT, can be considered as a descriptive model of risky decision making, but they were predicted by the configural weight models that were used to design the experiment.

Dissection of the Allais Paradox

Birnbaum (2004a) noted that constant consequence paradigm of Allais can be decomposed into three simpler properties: transitivity, coalescing, and restricted branch independence. *Transitivity* is the assumption that if one prefers A to B and prefers B to C, then one should prefer A to C. *Coalescing* is the assumption that if two branches of a gamble lead to the same consequence, they can be combined by adding their probabilities, without changing utility. For example, in Problem 1, coalescing implies that the gamble, A = ($1 million, 0.11; $0, 0.89), is identical in utility to A_s = ($1 million, 0.10; $1 million, 0.01; $0, 0.89), because A_s is one of the "split" forms of A, which is the "coalesced" form of A_s.

Restricted branch independence is the assumption that if two gambles with the same number of branches and same probability distribution over those branches have a common consequence on a branch, the common consequence can be changed to another value without altering the preference. For example, A_s = ($1 million, 0.10; $1 million, 0.01; $0, 0.89) is preferred to B_s = ($2 million, 0.10; $0, 0.01; $0, 0.89) if and only if C_s = ($1 million, 0.10; $1 million, 0.01; $1 million, 0.89) is preferred to D_s = ($2 million, 0.10; $0, 0.01; $1 million, 0.89), where the common branch of 0.89 to win $0 in the first choice has been changed to a common branch of 0.89 to win $1 million in the second.

If a person satisfied transitivity, coalescing, and restricted branch independence (all implied by EU), that person would not display the constant consequence paradox of Allais (Birnbaum, 2004a).

Consider the choice problems in Table 8.1. According to EU theory, the preference should be the same in all six choice problems, in the sense that A preferred to B if and only if (iff) A_s preferred to B_s, iff C_s preferred to D_s, iff C preferred to D, iff E_s preferred to F_s, and iff E preferred to F.

Original prospect theory (OPT), CPT, TAX, and EU all make different predictions for such a dissection of this Allais paradox, so one can compare all four theories by testing this "dissection" of the Allais paradox (Birnbaum, 2004a, 2007). Implications of the theories are shown in Table 8.2. As shown in Table 8.2, in EU theory, both restricted branch independence (columns in Table 8.2) and coalescing (rows in Table 8.2) are satisfied. OPT implies restricted branch independence and violates coalescing, to account for the Allais paradox. That is, OPT implies A_s is preferred to B_s, iff C_s is preferred to D_s and iff E_s is preferred to F_s. To explain an Allais paradox such as a reversal between the choice between A and B and between E and F, there must be a reversal either between choices of A versus B and A_s versus B_s or between the choices of E_s versus F_s and E versus F. OPT also had editing rules of combination and cancellation that imply coalescing and restricted branch independence, respectively, so OPT could mimic EU by invoking these editing rules, in which case the model does not predict Allais paradoxes.

Table 8.2 shows that, in contrast, CPT assumes coalescing and attributes the Allais paradox to violations of restricted branch independence; thus, A is preferred to B iff A_s is preferred to B_s, and E_s is preferred to F_s, iff E is preferred to F. If cancel-

Table 8.1 Six choice problems dissecting Allais paradox into tests of coalescing and restricted branch independence

No.	"Safe"	"Risky"
1	A: ($1 M, 0.11; $0, 0.89)	B: ($2 M, 0.10; $0, 0.90)
1s	A_s: ($1 M, 0.10; $1 M, 0.01; $0, 0.89)	B_s: ($2 M, 0.10; $0, 0.01; $0, 0.89)
2s	C_s: ($1 M, 0.10; $1 M, 0.01; $1 M, 0.89)	D_s: ($2 M, 0.10; $0, 0.01; $1 M, 0.89)
2	C: $1 M for sure	D: ($2 M, 0.10; $1 M, 0.89; $0, 0.01)
3s	E_s: ($1 M, 0.10; $1 M, 0.01; $2 M, 0.89)	F_s: ($2 M, 0.10; $0, 0.01; $2 M, 0.89)
3	E: ($2 M, 0.89; $1 M, 0.11)	F: ($2 M, 0.99; $0, 0.01)

According to coalescing, Choices No. 1 and 1s, 2 and 2s, and 3 and 3s are equivalent choice problems. According to restricted branch independence, Choices No. 1s, 2s, and 3s should all be either "safe" or they should all be "risky," but one should not switch systematically

Table 8.2 Comparison of decision theories

Restricted branch independence		
Coalescing	Satisfied	Violated
Satisfied	EUT	CPT
Violated	OPT	CWT

Notes: *EUT* expected utility theory, *CPT* cumulative prospect theory, *OPT* original prospect theory, *CWT* configural weight theory (TAX). The editing rules of combination and cancellation produce satisfaction of coalescing and restricted branch independence, respectively

lation was invoked, CPT could also mimic EU and would not predict Allais paradoxes.

Configural weight models such as TAX violate coalescing and with typical parameters, they often imply opposite violations of restricted branch independence from those required by CPT to account for the Allais paradoxes. According to this model, it should be possible to construct choice problems in which the Allais paradox would be reversed when the choices are presented in canonical split form. Canonical split form means that probabilities on ranked branches are equal, and the number of branches is minimal, as in choices A_s versus B_s and in E_s versus F_s.

Empirically, there are strong violations of both coalescing and of restricted branch independence, and the violations of restricted branch independence are indeed opposite the direction required by CPT to account for the Allais paradox (Birnbaum, 2004a, 2007, 2008b). Thus, EU and both versions of prospect theory can be rejected because they both cannot account for violations of both coalescing and restricted branch independence in the dissection of the Allais paradox.

A number of studies have now been completed testing between configural weight models and CPT investigating these and other critical behavioral properties that can be used to distinguish between these models. The results strongly refute both versions of prospect theory in favor of the predictions made by the configural models (Birnbaum, 2004a, 2004b, 2006, 2008b; Birnbaum & Bahra, 2012a).

Another criticism of CPT was developed, based on the idea that other process assumptions could be made to emulate its predictions for certain cases where the model had some success; the priority heuristic was constructed to fit previously published data.

Priority Heuristic and Relative Arguments

Brandstätter, Gigerenzer, and Hertwig (2006) based their priority heuristic model on the lexicographic semiorder that had been used by Tversky (1969) to describe intransitive preferences that Tversky believed he found in a small number of selected individuals.

According to the priority heuristic (PH), a person first compares lowest consequences of a gamble and chooses the gamble with the higher lowest consequence if they differ by more than 10% of the largest consequence in either gamble, rounded to the nearest prominent number. When the lowest consequences are not sufficiently different, the person chooses the gamble with the smaller probability to get the lowest consequence, if these differ by 0.1 or more. If the probabilities of the lowest consequences differed by less than 0.1, the person is theorized to next compare the highest prizes and choose by that criterion, if they differ sufficiently. When there are more than two branches and the first three comparisons yield no decision, the person next compares the probabilities to win the highest prize and decides on that basis alone, if there is any difference. And if all four criteria yield no decision, the person chooses randomly, without examining anything else. At each stage, the decision is based on only one reason, which is a contrast on one dimension. For example, comparing A = (\$5.00, 0.29; \$0, 0.71) versus C = (\$4.50, 0.38; \$0, 0.62), a person first compares the lowest outcomes, and since they are \$0 in both alternatives; next, she examines the probability to receive the lowest outcome, but since the difference is less than 0.10, she compares the highest consequences and decides that A is better than C.

A claim was made that the PH fit certain published choice data as well or better than EU, CPT, or TAX, but this claim was challenged and shown to hold only with selected data and only when certain assumptions are forced onto theories that do not make those assumptions (Birnbaum, 2008a); when other data sets were analyzed, the model performed very poorly, and when best-fit parameters are estimated from the data for all models, the PH with its best-fit parameters did not outperform CPT or TAX with their best-fit parameters.

The PH model had been constructed to account for the Allais paradoxes in original form, but it could not account for new examples such as the dissection of the Allais paradoxes (Birnbaum, 2004a), nor for violations of stochastic dominance (Birnbaum, 1999), nor for violations of restricted branch independence (Birnbaum & Navarrete, 1998). Although these phenomena had been published in the literature, they had not been included in the contest of fit that claimed high accuracy for the PH.

The PH implies systematic violations of transitivity that do not appear empirically. For example, with gambles *A* = ($5.00, 0.29; $0, 0.71), *C* = ($4.50, 0.38; $0, 0.62), and *E* = ($4.00, 0.46; $0, 0.54), the PH predicts that the majority should prefer *A* to *C*, and prefer *C* to *E*, and yet prefer *E* to *A*. But new studies by Birnbaum and Gutierrez (2007), Regenwetter, Dana, and Davis-Stober (2011), and Birnbaum and Bahra (2012b), among others, designed to test the predictions of the PH found that majority preferences did not show the predicted patterns of PH. In fact, the PH predicted only 30% of the modal choices correctly in Birnbaum and Gutierrez (a random coin toss would have correctly predicted 50%). PH model also does significantly worse than chance in predicting violations of restricted branch independence, because it predicts the opposite pattern of violations from what is observed in both group data and the majority of individuals analyzed separately (Birnbaum & Bahra, 2012a).

The PH implies that attributes or dimensions of a stimulus do not combine, nor do they interact, but experimental tests of combination and interaction showed evidence that people integrate information between dimensions and that the dimensions interact. For example, consider the following two choice problems:

Problem 6: *X* = ($100, 0.9; $5, 0.1)

Y = ($50, 0.9; $20, 0.1)

Problem 7: *X'* = ($100, 0.1; $5, 0.9)

Y' = ($50, 0.1; $20, 0.9)

According to the PH, a person should choose *Y* and *Y'* because the probabilities are the same and the lowest consequences are better by the same amount in both gambles. According to another lexicographic semiorder, a person might choose *X* and *X'*, if they examined the highest consequences first. Because the probabilities are the same in both gambles within each choice, probability should not make any difference in these models. However, most people choose *X* over *Y* in Problem 6 and choose *Y'* over *X'* in Problem 7, contrary to any lexicographic semiorder model. These violations also contradict other similarity models that decide by comparing contrasts between components but do not postulate that components interact (Birnbaum, 2008c, 2010).

The perceived relative arguments model (Loomes, 2010), like the priority heuristic and regret theory (Loomes, Starmer, & Sugden, 1991; Loomes & Sugden, 1982), can also violate transitivity. The Loomes (2010) model assumes that people make choices by combining contrasts between the components, so it differs from the PH. However, empirical studies of predicted intransitivity by regret theory and perceived relative arguments model have not confirmed its predictions, and this model also fails to account for violations of restricted branch independence (Birnbaum & Diecidue, 2015).

In principle, violations of transitivity, if substantial and systematic, would rule out a large class of models that includes EV, EU, CPT, and TAX. Therefore, it would be extremely important to know if stimuli can be found that produce predictable, systematic violations of transitivity. But Birnbaum and Diecidue (2015) noted that specific tests for intransitivity have not shown convincing evidence favoring either the PH, regret theory, or similarity models over the family of transitive models.

Decisions from Description and from Experience

Much of the research and theory presented to this point has been based on cases where a decision-maker makes a decision based on descriptions of the consequences and their likelihoods. This paradigm matches many real-life situations where people make decisions without previous experience. However, in some cases, people make repeated decisions and can use their experience to revise beliefs about the likelihoods and utilities of the consequences.

Hertwig, Barron, Weber, and Erev (2004) contrast two paradigms for decision making research. The first method asks people to make a single decision based on descriptions of the relevant chances and consequences, and the second method involves learning of probabilities based on experience with a sequence of events representing some unknown stochastic process. A sick person deciding which of two medical treatments to choose seems to match the first method, whereas an experienced person deciding what to order from a frequently visited restaurant seems to illustrate the second. With description, many people say they prefer a small chance at a large prize to a sure thing with the same expected value. For example, many people prefer M = ($100, 0.01; $0, 0.99) over N = $1 for sure, based on a description.

Such risk-seeking behavior for small probabilities to win positive consequences is consistent with OPT, CPT, and TAX, given typical parameters. However, when people are asked to sample from the two options, and then asked to make a choice, they often choose the safe option over the risky gamble. Hertwig et al. (2004) argued that perhaps different theories of decision making might be required for these two types of situation. They note that learning and perception of probabilities might be overly influenced by the particular sequence of events.

However, Fox and Hadar (2006) noted that from the perspective of experience, some people who drew small samples might experience M as "$0 always occurs" (since the unlikely event of $100 might never occur in a small sample); in contrast, they experience N as "always pays $1"; they never experience the population, so subjectively, the choice was between always $0 and always $1. In many studies done in this field, sampling is left to the participant and to chance, so the experience has not been constrained to match the description.

Glöckner, Hilbig, Henninger, and Fiedler (2016) present a current review of the literature on description versus experience, a reanalysis of earlier studies, and new experiments designed to disentangle different interpretations. They conclude that sampling and regression effects are important components of the previous studies, but they argue that other factors (such as uncertainty) play roles as well. For example, how does one learn from a brief experience that something is a "sure thing?"

When one hears the description, "you win \$50 no matter what color you draw from the urn," it denotes a sure thing. This case is different from the situation of 15 trials that yield only \$50 prizes. There is still the chance that other prizes might occur that have not yet been experienced. One factor that has not yet been addressed in this literature on experience versus description that has been considered by some in the description literature is the role of error or variability of response to producing choice behavior.

Models of Error or Variability

When a person is presented the same choice problem on two occasions, the same person will often make a different choice responses on the two trials. For example, consider the next two choice problems:

Problem 8: *R* = (\$98, 0.10; \$2, 0.90)

S = (\$40, 0.20; \$2, 0.80)

Problem 9: *R'* = (\$98, 0.90; \$2, 0.10)

S' = (\$98, 0.80; \$40, 0.20)

Problems 8 and 9 were included, separated by a number of intervening trials, among a list of 31 choice problems. Following a brief intervening task of about 10 min, the same people were asked to respond to the same choice problems a second time. It was found that the same people made different responses 20% of the time on Problem 9, and 31% reversed preferences on two presentations of Problem 8. According to EU, a person should prefer *R* over *S* if and only if she prefers *R'* over *S'*. But if the same person can change responses when Problem 8 is presented twice, should we be surprised if that same person made different responses on Problems 8 and 9?

In the past, researchers argued that if significantly more people chose *R* in Problem 8 and *S'* in Problem 9 than the number who made the opposite pattern of reversal (*S* and *R'*), then the "significant" difference meant one should reject EU. However, it has recently been shown that if different choice problems have different rates of error, then such asymmetry of reversals could occur even if EU held true. The idea that inherent variability or errors in choice might produce some or all of the apparent violations in tests of the Allais paradoxes (or of other behavioral properties such as transitivity) has been an important focus of recent research (Birnbaum, 2013; Birnbaum & Bahra, 2012a; Carbone & Hey, 2000; Loomes, 2005; Regenwetter et al., 2011; Wilcox, 2008).

A family of models known as "true and error" models has been developed, based on the idea that one can estimate the error component from preference reversals by the same person to the same choice problems within a brief session. These models allow that a person's "true" preferences may have variability between sessions (blocks of trials), due to such factors as changing parameters or changing models, and they allow separation of such variability due to a mixture of models from variability produced by "error" that produces reversals within a session. They allow each choice problem to have a different rate of error, and they allow different people to have differing amounts of noise or unreliability in their responses.

When these models have been applied to repeated judgments, it has been found that the violations of EU, as in the Allais paradoxes; violations of CPT, as in the "new paradoxes"; and violations of the priority heuristic, as in the tests of interactive independence, cannot be attributed to this type of error (Birnbaum, 2008b, 2008c, 2010). On the other hand, violations of transitivity have been found to be of low frequency when the inherent variability of the data is fit by the true and error model.

Concluding Comments

The field of risky decision making is one of the oldest topics in behavioral science and has influenced both psychology and economics. Over the years, new models have been developed, and new evidence has accumulated to refute some theories in favor of others. When new evidence violates a currently popular model, the findings are often called "paradoxes" or "anomalies." Data have shown that EV, EU, SWU, OPT, and CPT can be rejected based on violations of critical properties. Intransitive models such as regret theory, lexicographic semiorders, and the priority heuristic have not yet been able to show where to find the predicted intransitive preference cycles nor have they been successful in predicting results of new experiments designed to test them. Configural weight models, such as TAX, remain the best account of the major phenomena, but as new research is conducted, it seems likely that more accurate and elegant models can be developed. As new theories are developed, new tests are designed, and new information is gained about how people deal with risk in making decisions.

References

Allais, M. (1953). Fondements d'une Théorie Positive des Choix Comportant un Risque et Critique des Postulats et Axiomes de l'Ecole Américaine. *Econometrica, 21*, 503–546.

Allais, M. (1979). The foundations of a positive theory of choice involving risk and a criticism of the postulates and axioms of the American school (1952). In M. Allais & O. Hagen (Eds.), *Expected utility hypotheses and the Allais Paradox* (pp. 27–145). Dordrecht: Reidel/Springer.

Bernoulli, D. (1954). Exposition of a new theory on the measurement of risk. *Econometrica, 22*, 23–36 (English translation of 1738 paper).

Birnbaum, M. H. (1974). The nonadditivity of personality impressions. *Journal of Experimental Psychology Monograph, 102*, 543–561.

Birnbaum, M. H. (2004a). Causes of Allais common consequence paradoxes: An experimental dissection. *Journal of Mathematical Psychology, 48*, 87–106.

Birnbaum, M. H. (2004b). Tests of rank-dependent utility and cumulative prospect theory in gambles represented by natural frequencies: Effects of format, event framing, and branch splitting. *Organizational Behavior and Human Decision Processes, 95*, 40–65.

Birnbaum, M. H. (2006). Evidence against prospect theories in gambles with positive, negative, and mixed consequences. *Journal of Economic Psychology, 27*, 737–761.

Birnbaum, M. H. (2007). Tests of branch splitting and branch-splitting independence in Allais paradoxes with positive and mixed consequences. *Organizational Behavior and Human Decision Processes, 102*, 154–173.

Birnbaum, M. H. (2008a). Evaluation of the priority heuristic as a descriptive model of risky decision making: Comment on Brandstätter, Gigerenzer, and Hertwig (2006). *Psychological Review, 115*, 253–260.

Birnbaum, M. H. (2008b). New paradoxes of risky decision making. *Psychological Review, 115*, 463–501.

Birnbaum, M. H. (2008c). New tests of cumulative prospect theory and the priority heuristic: Probability-outcome tradeoff with branch splitting. *Judgment and Decision making, 3*, 304–316.

Birnbaum, M. H. (2010). Testing lexicographic semiorders as models of decision making: Priority dominance, integration, interaction, and transitivity. *Journal of Mathematical Psychology, 54*, 363–386.

Birnbaum, M. H. (2013). True-and-error models violate independence and yet they are testable. *Judgment and Decision making, 8*, 717–737.

Birnbaum, M. H., & Bahra, J. P. (2012a). Separating response variability from structural inconsistency to test models of risky decision making. *Judgment and Decision making, 7*, 402–426.

Birnbaum, M. H., & Bahra, J. P. (2012b). Testing transitivity of preferences in individuals using linked designs. *Judgment and Decision making, 7*, 524–567.

Birnbaum, M. H., & Diecidue, E. (2015). Testing a class of models that includes majority rule and regret theories: Transitivity, recycling, and restricted branch independence. *Decision, 2*, 145–190.

Birnbaum, M. H., & Gutierrez, R. J. (2007). Testing for intransitivity of preferences predicted by a lexicographic semiorder. *Organizational Behavior and Human Decision Processes, 104*, 97–112.

Birnbaum, M. H., & Navarrete, J. (1998). Testing descriptive utility theories: Violations of stochastic dominance and cumulative independence. *Journal of Risk and Uncertainty, 17*, 49–78.

Birnbaum, M. H., & Stegner, S. E. (1979). Source credibility in social judgment: Bias, expertise, and the judge's point of view. *Journal of Personality and Social Psychology, 37*, 48–74.

Brandstätter, E., Gigerenzer, G., & Hertwig, R. (2006). The priority heuristic: Choices without tradeoffs. *Psychological Review, 113*, 409–432.

Birnbaum, M. H. (1999). Paradoxes of Allais, stochastic dominance, and decision weights. In J. Shanteau, B. A. Mellers, & D. A. Schum (Eds.), *Decision science and technology: Reflections on the contributions of Ward Edwards* (pp. 27-52). Norwell, MA: Kluwer Academic Publishers.

Carbone, E., & Hey, J. D. (2000). Which error story is best? *Journal of Risk and Uncertainty, 20*, 161–176.

Edwards, W. (1954). The theory of decision making. *Psychological Bulletin, 51*, 380–417.

Edwards, W. (1962). Subjective probabilities inferred from decisions. *Psychological Review, 69*, 109–135.

Fox, C. R., & Hadar, L. (2006). "Decisions from experience" = sampling error + prospect theory: Reconsidering Hertwig, Barron, Weber & Erev (2004). *Judgment and Decision making, 1*, 159–161.

Glöckner, A., Hilbig, B., Henninger, F., & Fiedler, S. (2016). The reversed description-experience gap: Disentangling sources of presentation format effects in risky choice. *Journal of Experimental Psychology: General, 145*, 486–508.

Hertwig, R., Barron, G., Weber, E. U., & Erev, I. (2004). Decisions from experience and the effect of rare events in risky choice. *Psychological Science, 15*, 534–539.

Kahneman, D. (2003). Experiences of collaborative research. *American Psychologist, 58*, 723–730.
Kahneman, D., & Tversky, A. (1979). Prospect theory: An analysis of decision under risk. *Econometrica, 47*, 263–291.
Loomes, G. (2005). Modeling the stochastic component of behavior in experiments: Some issues for interpretation of data. *Experimental Economics, 8*, 301–323.
Loomes, G. (2010). Modeling choice and valuation in decision experiments. *Psychological Review, 117*, 902–924.
Loomes, G., Starmer, C., & Sugden, R. (1991). Observing violations of transitivity by experimental methods. *Econometrica, 59*, 425–440.
Loomes, G., & Sugden, R. (1982). Regret theory: An alternative theory of rational choice under uncertainty. *The Economic Journal, 92*, 805–824.
Luce, R. D. (2000). *Utility of gains and losses: Measurement-theoretical and experimental approaches*. Mahwah, NJ: Lawrence Erlbaum Associates.
Luce, R. D., & Fishburn, P. C. (1991). Rank- and sign-dependent linear utility models for finite first order gambles. *Journal of Risk and Uncertainty, 4*, 29–59.
Luce, R. D., & Fishburn, P. C. (1995). A note on deriving rank-dependent utility using additive joint receipts. *Journal of Risk and Uncertainty, 11*, 5–16.
Markowitz, H. (1952). The utility of wealth. *Journal of Political Economy, 60*, 151–158.
Marley, A. A. J., & Luce, R. D. (2005). Independence properties vis-à-vis several utility representations. *Theory and Decision, 58*, 77–143.
Quiggin, J. (1985). Subjective utility, anticipated utility, and the Allais paradox. *Organizational Behavior and Human Decision Processes, 35*, 94–101.
Quiggin, J. (1993). *Generalized expected utility theory: The rank dependent model*. Boston: Kluwer.
Regenwetter, M., Dana, J., & Davis-Stober, C. P. (2011). Transitivity of preferences. *Psychological Review, 118*, 42–56.
Tversky, A., & Kahneman, D. (1992). Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty, 5*, 297–323.
Tversky, A. (1969). Intransitivity of preferences. *Psychological Review,* 76, 31–48.
Von Neumann, J., & Morgenstern, O. (1947). *Theory of games and economic behavior* (2nd ed.). Princeton: Princeton University Press https://archive.org/details/theoryofgamesand030098mbp
Wilcox, N. (2008). Stochastic models for binary discrete choice under risk: A critical primer and econometric comparison. In J. C. Cox & G. W. Harrison (Eds.), *Research in experimental economics vol. 12: Risk aversion in experiments* (pp. 197–292). Bingley: Emerald.